

Yang-Mills Instantons from Gravitational Instantons

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ABSTRACT

We show that every gravitational instantons are $SU(2)$ Yang-Mills instantons on a Ricci-flat four manifold although the reverse is not necessarily true. It is shown that gravitational instantons satisfy exactly the same self-duality equation of $SU(2)$ Yang-Mills instantons on the Ricci-flat manifold determined by the gravitational instantons themselves. We explicitly check the correspondence with several examples and discuss their topological properties.

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1 Introduction

An instanton in gauge theories is a topologically nontrivial solution described by a self-dual or anti-self-dual connection with a finite action. Such instantons play an important role in the nonperturbative dynamics of gauge theories, in particular, to understand the vacuum structure of quantum field theories [1]. One of the most powerful uses of instantons in recent years is in the analysis of strongly coupled gauge dynamics where they play a key role in unraveling the plexus of entangled dualities that relates different theories. One of the highlights is the remarkable theory of Seiberg and Witten [2] which determines the low-energy behavior of $\mathcal{N} = 2$ supersymmetric gauge theories exactly. In $\mathcal{N} = 2$ supersymmetric gauge theories, the instantons lead to quantum corrections for the metric on the moduli space of vacua.

A semi-classical evaluation of the path integral requires us to find the complete set of finite-action configurations which minimize the Euclidean action. In pure Yang-Mills theory, the complete set of self-dual gauge fields of arbitrary topological charge k can be obtained by solving some quadratic matrix equations, known as the Atiyah-Drinfeld-Hitchin-Manin (ADHM) equations [3], which are a set of nonlinear algebraic equations constraining a matrix of moduli parameters. It can be shown [4] that the functional integral in the semi-classical approximation reduces to an integral over the instanton moduli space in each instanton sector. In principle the low-energy effective action can also be calculated from first principles via conventional semi-classical methods using instantons.

It is known that the instanton calculus in supersymmetric theories is fully controllable when the theories are weakly coupled. This leads to the idea of testing the Seiberg-Witten theory by calculating the instanton effects and comparing these expressions with those extracted from the Seiberg-Witten curve. For reviews, see, for example, [4, 5]. Since the integral over a generic instanton moduli space is too complicated to be done directly, it was fully accomplished only recently by using the localization technique and considering the resolution of the instanton moduli space via the ADHM construction relevant to a noncommutative gauge theory [6]. It has been checked [7] that the results computed using the method of localization perfectly agree with the Seiberg-Witten solution for $\mathcal{N} = 2$ supersymmetric gauge theories.

On the mathematical side, instantons lie at the heart of the recent works on the topology of four-manifolds [8]. In particular, Donaldson used the moduli space of instantons over a differentiable four-manifold to construct topological invariants of the four-manifold and showed that the moduli spaces of instantons often carry nontrivial and surprising information about the background manifold.

One would like to extend the path integral approach to include gravitation. Although the Euclidean gravitational action is not positive-definite even for real positive-definite metrics, one can evaluate the functional integral by first looking for non-singular stationary points of the action functional and expand about them. Such critical points are finite action solutions to the classical field equations called “gravitational instantons,” the gravitational analogue of Yang-Mills instantons [9]. These are defined as complete, non-singular, and positive-definite metrics which are self-dual or anti-self-dual

metrics of vacuum Einstein equations [10]. One can show [11] that the self-dual or anti-self-dual metrics are local minima of the action among metrics with zero scalar curvature.

In general relativity, the Lorentz group appears as the structure group acting on orthonormal frames in the tangent space of a Riemannian manifold M [12]. Under a local Lorentz transformation which is the orthogonal rotation group $O(4)$, a matrix-valued spin connection $\omega^A_B = \omega_M^A{}_B dx^M$ plays a role of gauge fields in $O(4)$ gauge theory. From the $O(4)$ gauge theory point of view, the Riemann curvature tensors precisely correspond to the field strengths of the $O(4)$ gauge fields $\omega_M^A{}_B$. (More details will be explained in Sect. 3.) Since the group $O(4)$ is a direct product of normal subgroups $SU(2)_L$ and $SU(2)_R$, i.e. $O(4) = SU(2)_L \times SU(2)_R$, the four-dimensional Euclidean gravity, when formulated as the $O(4)$ gauge theory, will basically be two copies of $SU(2)$ gauge theories.

As we summarized above, Yang-Mills instantons are important to determine the vacuum structure of quantum field theories and the ADHM construction provides a description of all instantons on \mathbb{R}^4 in terms of algebraic data. One would expect that gravitational instantons play a similar substantial role in quantum gravity although the quantum aspect of general relativity has encountered long-standing difficulties because there is hardly any common ground between general relativity and quantum mechanics. The well-known divergences in a quantum theory of gravity suggest that a field theory of gravity like as Einstein's general relativity is a purely low-energy or large-distance approximation to some more fundamental theory. Therefore, the gauge theory formulation of gravity may be helpful to glimpse some basic structures of such a fundamental theory because nonperturbative and quantum aspects about gauge theories are relatively well-known.

Whereas gravity is different from gauge theory in several marked ways, underlying mathematical structures are very similar to each other in many ways [10]. See, for example, the Table 1 in [13]. It was shown [14, 15, 16, 17] that certain classes of gravitational instantons such as the asymptotically locally Euclidean (ALE) and the asymptotically locally flat (ALF) hyper-Kähler four-manifolds can be constructed as a hyper-Kähler quotient of a finite-dimensional Euclidean space.¹ This construction is actually akin to the ADHM construction of Yang-Mills instantons on \mathbb{R}^4 [3] and has a natural interpretation in terms of D-branes in string theory. Moreover, the hyper-Kähler quotient construction of Yang-Mills instantons on an ALE or ALF space [19, 20, 21, 22] is a natural generalization of the original ADHM construction of instantons on flat space. The study of Yang-Mills theories on a curved manifold has recently received renewed attention because they are involved with effective field theories of D-brane and NS5-brane configurations [23, 24].

Now our motivation of this paper has surfaced. In this paper and its sequels, we wish to go beyond a mere formal analogy between gravity and gauge theory and try to answer to the following questions:

- A. What is the precise relation between gravity and gauge theory variables?
- B. How much are they parallel?
- C. How is the topology of a Riemannian manifold M encoded into gauge fields?

¹A general construction of essentially all known deformation classes of gravitational instantons was recently reported in [18].

D. What are crucial differences?

E. Can it be applied to examine a quantum nature of gravity?

The paper is organized as follows. In Section 2, we will summarize Yang-Mills instantons on a curved four-manifold to set our notation and explain why Yang-Mills instantons on a Ricci-flat manifold is a solution of the coupled equations in Einstein-Yang-Mills theory.

In Section 3, we will employ the decomposition in [25] to explicitly realize that the Lorentz group $O(4)$ is a direct product of normal subgroups $SU(2)_L$ and $SU(2)_R$, i.e. $O(4) = SU(2)_L \times SU(2)_R$.² It is then easy to show [27] that the four-dimensional Euclidean gravity, when formulated as the $O(4)$ gauge theory, will basically be two copies of $SU(2)$ gauge theories. In particular, it can be shown that one of $SU(2)$'s decouples from the theory when considering self-dual or anti-self-dual metrics called gravitational instantons. As a result, one can show that gravitational instantons satisfy exactly the same self-duality equation of $SU(2)$ Yang-Mills instantons on the Ricci-flat manifold determined by the gravitational instantons themselves. Therefore, every gravitational instantons can be interpreted as self-gravitating $SU(2)$ Yang-Mills instantons although the reverse is not necessarily true. This provides a powerful method to find a particular class of Yang-Mills instantons on a general self-dual four manifold.

In Section 4, we will elucidate with explicit examples how it is always possible to find Yang-Mills instantons on a Ricci-flat manifold M using the prescription in Section 3 whenever a gravitational instanton solution is given. Our method vividly realizes the Charap-Duff prescription [27] for $SU(2)$ Yang-Mills instantons on a Ricci-flat manifold (see also [28]). We will easily reproduce already known solutions in literatures [29, 30, 31, 32] in this way and also find new Yang-Mills instantons as a byproduct.

In Section 5, some issues about topological invariants for Riemannian manifolds will be discussed. In the gravity side, there are two topological invariants [10] known as the Euler characteristic $\chi(M)$ and the Hirzebruch signature $\tau(M)$, while, in the gauge theory side, there is a unique topological invariant up to a boundary term given by the Chern class of gauge bundle. The correspondence between gravitational and Yang-Mills instantons then implies that the two topological invariants for gravitational instantons should be related to each other. We conjecture a possible relation between $\chi(M)$ and $\tau(M)$ by inspecting several known results in literatures [33, 34, 35].

In Section 6, we draw our conclusions and discuss open issues for future works.

Finally, we set up our index notation which is especially useful for the explicit calculation in Section 4; otherwise diverse spaces we are considering would lead to some confusions.

Index notation We employ the following index convention throughout the paper:

- $M, N, P, Q, \dots = 1, \dots, 4$: world (curved space) indices,
- $A, B, C, D, \dots = \hat{1}, \dots, \hat{4}$: frame (tangent space) indices,
- $i, j, k, l, \dots = 1, 2, 3$: three-dimensional world indices,
- $\hat{i}, \hat{j}, \hat{k}, \hat{l}, \dots = \hat{1}, \hat{2}, \hat{3}$: three-dimensional frame indices,

²See also [26] for geometric aspects of the decomposition according to the group structure $O(4) = SU(2)_L \times SU(2)_R$.

- $a, b, c, d, \dots = \dot{1}, \dot{2}, \dot{3} : SU(2)$ Lie algebra indices.

2 Yang-Mills Instantons on Riemannian Manifold

Consider a curved four-manifold M whose metric is given by

$$ds^2 = g_{MN}(x) dx^M dx^N. \quad (2.1)$$

Let $\pi : E \rightarrow M$ be an $SU(2)$ bundle over M whose curvature is defined by

$$\begin{aligned} F &= dA + A \wedge A = \frac{1}{2} F_{MN}(x) dx^M \wedge dx^N \\ &= \frac{1}{2} \left(\partial_M A_N - \partial_N A_M + [A_M, A_N] \right) dx^M \wedge dx^N \end{aligned} \quad (2.2)$$

where $A = A_M^a(x) T^a dx^M$ is a connection one-form of the vector bundle E . The generators T^a of $SU(2)$ Lie algebra satisfy the relation

$$[T^a, T^b] = -2\varepsilon^{abc} T^c \quad (2.3)$$

where we choose an unconventional normalization $\text{Tr} T^a T^b = -4\delta^{ab}$ for later purpose.

Let us introduce at each spacetime point in M a local frame of reference in the form of four linearly independent vectors (vierbeins or tetrads) $E_A = E_A^M \partial_M \in \Gamma(TM)$ which are chosen to be orthonormal, i.e., $E_A \cdot E_B = \delta_{AB}$. The frame basis $\{E_A\}$ defines a dual basis $E^A = E_M^A dx^M \in \Gamma(T^*M)$ by a natural pairing

$$\langle E^A, E_B \rangle = \delta_B^A. \quad (2.4)$$

The above pairing leads to the relation $E_M^A E_B^M = \delta_B^A$. In terms of the non-coordinate (anholonomic) basis in $\Gamma(TM)$ or $\Gamma(T^*M)$, the metric (2.1) can be written as

$$\begin{aligned} ds^2 &= \delta_{AB} E^A \otimes E^B = \delta_{AB} E_M^A E_N^B dx^M \otimes dx^N \\ &\equiv g_{MN}(x) dx^M \otimes dx^N \end{aligned} \quad (2.5)$$

or

$$\begin{aligned} \left(\frac{\partial}{\partial s} \right)^2 &= \delta^{AB} E_A \otimes E_B = \delta^{AB} E_A^M E_B^N \partial_M \otimes \partial_N \\ &\equiv g^{MN}(x) \partial_M \otimes \partial_N. \end{aligned} \quad (2.6)$$

Using the form language where $d = dx^M \partial_M = E^A E_A$ and $A = A_M dx^M = A_A E^A$, the field strength (2.2) of $SU(2)$ gauge fields in the non-coordinate basis takes the form

$$\begin{aligned} F &= dA + A \wedge A = \frac{1}{2} F_{AB} E^A \wedge E^B \\ &= \frac{1}{2} \left(E_A A_B - E_B A_A + [A_A, A_B] + f_{AB}^C A_C \right) E^A \wedge E^B \end{aligned} \quad (2.7)$$

where we used the structure equation

$$dE^A = \frac{1}{2}f_{BC}^A E^B \wedge E^C. \quad (2.8)$$

The frame basis $E_A = E_A^M \partial_M \in \Gamma(TM)$ satisfies the Lie algebra under the Lie bracket

$$[E_A, E_B] = -f_{AB}^C E_C \quad (2.9)$$

where

$$f_{ABC} = E_A^M E_B^N (\partial_M E_{NC} - \partial_N E_{MC}) \quad (2.10)$$

are the structure functions in (2.8).

Consider $SU(2)$ Yang-Mills theory defined on the Riemannian manifold (2.1) whose action is given by

$$S_{YM} = -\frac{1}{16g_{YM}^2} \int_M d^4x \sqrt{g} g^{MP} g^{NQ} \text{Tr} F_{MN} F_{PQ}. \quad (2.11)$$

The self-duality equation for the action (2.11) can be derived by observing the following identity

$$S_{YM} = -\frac{1}{32g_{YM}^2} \int_M d^4x \sqrt{g} \text{Tr} \left(F_{MN} \mp \frac{1}{2} \frac{\varepsilon^{RSPQ}}{\sqrt{g}} g_{MR} g_{NS} F_{PQ} \right)^2 \mp \frac{1}{32g_{YM}^2} \int_M d^4x \varepsilon^{MNPQ} \text{Tr} F_{MN} F_{PQ}, \quad (2.12)$$

where ε^{MNPQ} is the metric independent Levi-Civita symbol with $\varepsilon^{1234} = 1$. Note that the second term in Eq. (2.12) is a topological term (total derivative) and so does not affect the equations of motion. Because the first term in Eq. (2.12) is positive-definite, the minimum of the action (2.11) can be achieved by the self-dual gauge fields (instantons) satisfying

$$F_{MN} = \pm \frac{1}{2} \frac{\varepsilon^{RSPQ}}{\sqrt{g}} g_{MR} g_{NS} F_{PQ}. \quad (2.13)$$

In the non-coordinate basis, the self-duality equation (2.13) can be written as the form

$$F_{AB} = \pm \frac{1}{2} \varepsilon_{AB}^{CD} F_{CD} \quad (2.14)$$

with the field strength $F_{AB} = E_A^M E_B^N F_{MN}$ in (2.7).

It is easy to check that the $SU(2)$ instantons defined by (2.13) automatically satisfy the equations of motion

$$g^{MN} D_M F_{NP} = 0 \quad (2.15)$$

because we have the following relation from the self-duality (2.13)

$$g^{MN} D_M F_{NP} = \mp \frac{1}{2} g_{PQ} \frac{\varepsilon^{QMNR}}{\sqrt{g}} D_M F_{NR} = 0 \quad (2.16)$$

where we used the Bianchi identity for the $SU(2)$ curvature (2.2), i.e.

$$\varepsilon^{MNPQ} D_N F_{PQ} = 0. \quad (2.17)$$

The covariant derivative in (2.15) is with respect to both the Yang-Mills and gravitational connections, i.e.

$$D_M F_{NP} = \partial_M F_{NP} - \Gamma_{MN}^Q F_{QP} - \Gamma_{MP}^Q F_{NQ} + [A_M, F_{NP}], \quad (2.18)$$

where Γ_{MN}^P is the Levi-Civita connection.

Now the problem we pose here is how to construct instanton solutions satisfying (2.13). Several questions immediately arise. Is it possible to find an instanton solution satisfying (2.13) on an arbitrary Riemannian manifold ? Or is there any constraint on the background manifold for the existence of Yang-Mills instantons ? What is the moduli space of $SU(2)$ instantons defined on a given four-manifold M ?

We think the above questions are still open. Nevertheless, there are several examples on Yang-Mills instantons defined on a curved four-manifold. For example, the famous ADHM construction on \mathbb{S}^4 [3], Yang-Mills instantons on \mathbb{CP}^2 [36], $\mathbb{H} \times \mathbb{S}^2$ (\mathbb{H} = Poincaré half-plane) [37], ALE [19, 20] and ALF spaces [21, 22]. Also many other solutions have been constructed so far [38, 39, 40]. See, for example, [41, 42] for a review and references therein. In particular, Taubes proved [43] that all compact oriented four-manifolds admit nontrivial instantons. But recently it was shown [44] that there exists a noncompact four-manifold having no nontrivial instanton. So far, we do not have a general description à la ADHM of all instantons satisfying the self-duality (2.13).

We will show that a large class of Yang-Mills instantons satisfying (2.13) or (2.14) can be solved by gravitational instantons. To be precise, we will show that every gravitational instantons satisfy the self-duality equation (2.13) for $SU(2)$ gauge fields on a Riemannian manifold defined by the gravitational instanton itself. To prepare our setup, let us consider the case when $SU(2)$ Yang-Mills and gravitational fields are both dynamically active. The total action is defined by

$$S = S_{YM} + S_G \quad (2.19)$$

where the Yang-Mills action S_{YM} is given by (2.11) and the gravitational action is given by

$$S_G = \frac{1}{16\pi G} \int_M d^4x \sqrt{g} R + \text{surface terms}. \quad (2.20)$$

The gravitational field equations read as

$$R_{MN} - \frac{1}{2} g_{MN} R = 8\pi G T_{MN} \quad (2.21)$$

with

$$T_{MN} = \frac{1}{4g_{YM}^2} \text{Tr} \left(g^{PQ} F_{MP} F_{NQ} - \frac{1}{4} g_{MN} F_{PQ} F^{PQ} \right). \quad (2.22)$$

For an instanton solution satisfying Eq.(2.13), the energy-momentum tensor (2.22) identically vanishes, i.e. $T_{MN} = 0$ and then Eq.(2.21) enforces the vacuum Einstein equations

$$R_{MN} = 0. \quad (2.23)$$

Conversely, the reason that (anti-)self-dual Yang-Mills fields do not spoil Ricci-flatness of a manifold is due to the vanishing of the Euclidean energy-momentum tensor (2.22). Our interest is to solve the coupled equations (2.13) and (2.21) simultaneously. Therefore, the four-manifold M in Eq.(2.13) should be Ricci-flat, i.e., satisfying the vacuum Einstein equations (2.23).

3 Gravitational Instantons

Under local frame rotations in $O(4)$, the vectors transform according to

$$\begin{aligned} E_A(x) &\rightarrow E'_A(x) = E_B(x)\Lambda^B{}_A(x), \\ E^A(x) &\rightarrow E^{A'}(x) = \Lambda^A{}_B(x)E^B(x) \end{aligned} \quad (3.1)$$

where $\Lambda^A{}_B(x) \in O(4)$. The spin connections $\omega_M(x)$ then constitute gauge fields with respect to the local $O(4)$ rotations

$$\omega_M \rightarrow \Lambda \omega_M \Lambda^{-1} + \Lambda \partial_M \Lambda^{-1} \quad (3.2)$$

and the covariant derivative is defined by

$$\begin{aligned} D_M E_A &= \partial_M E_A - \omega_M^B{}_A E_B, \\ D_M E^A &= \partial_M E^A + \omega_M^A{}_B E^B. \end{aligned} \quad (3.3)$$

The connection one-forms $\omega^A{}_B = \omega_M^A{}_B dx^M$ satisfy the Cartan's structure equations [12],

$$T^A = dE^A + \omega^A{}_B \wedge E^B, \quad (3.4)$$

$$R^A{}_B = d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B, \quad (3.5)$$

where T^A are the torsion two-forms and $R^A{}_B$ are the curvature two-forms. In terms of local coordinates, they are given by

$$T_{MN}{}^A = \partial_M E_N^A - \partial_N E_M^A + \omega_M^A{}_B E_N^B - \omega_N^A{}_B E_M^B, \quad (3.6)$$

$$R_{MN}{}^A{}_B = \partial_M \omega_N^A{}_B - \partial_N \omega_M^A{}_B + \omega_M^A{}_C \omega_N^C{}_B - \omega_N^A{}_C \omega_M^C{}_B. \quad (3.7)$$

Now we impose the torsion free condition, $T_{MN}{}^A = D_M E_N^A - D_N E_M^A = 0$, to recover the standard content of general relativity, which eliminates ω_M as an independent variable, i.e.,

$$\begin{aligned} \omega_{ABC} &= E_A^M \omega_{MBC} = \frac{1}{2}(f_{ABC} - f_{BCA} + f_{CAB}) \\ &= -\omega_{ACB} \end{aligned} \quad (3.8)$$

where f_{ABC} are the structure functions given by (2.10). The spin connection (3.8) is related to the Levi-Civita connection as follows

$$\Gamma_{MN}{}^P = \omega_M^A{}_B E_A^P E_N^B + E_A^P \partial_M E_N^A. \quad (3.9)$$

Since the spin connection ω_{MAB} and the curvature tensor R_{MNAB} are antisymmetric on the AB index pair, one can decompose them into a self-dual part and an anti-self-dual part as follows [25, 26]

$$\omega_{MAB} \equiv A_M^{(+a)} \eta_{AB}^a + A_M^{(-a)} \bar{\eta}_{AB}^a, \quad (3.10)$$

$$R_{MNAB} \equiv F_{MN}^{(+a)} \eta_{AB}^a + F_{MN}^{(-a)} \bar{\eta}_{AB}^a, \quad (3.11)$$

where the 4×4 matrices η_{AB}^a and $\bar{\eta}_{AB}^a$ for $a = \hat{1}, \hat{2}, \hat{3}$ are 't Hooft symbols defined by

$$\begin{aligned} \bar{\eta}_{\hat{i}\hat{j}}^a &= \eta_{\hat{i}\hat{j}}^a = \varepsilon_{a\hat{i}\hat{j}}, & \hat{i}, \hat{j} &\in \{\hat{1}, \hat{2}, \hat{3}\}, \\ \bar{\eta}_{\hat{4}\hat{i}}^a &= \eta_{\hat{i}\hat{4}}^a = \delta_{a\hat{i}}. \end{aligned} \quad (3.12)$$

Note that the 't Hooft matrices intertwine the group structure of the index a with the spacetime structure of the indices A, B . We list some useful identities of the 't Hooft tensors [25, 26]

$$\eta_{AB}^{(\pm)a} = \pm \frac{1}{2} \varepsilon_{AB}^{CD} \eta_{CD}^{(\pm)a}, \quad (3.13)$$

$$\eta_{AB}^{(\pm)a} \eta_{CD}^{(\pm)a} = \delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC} \pm \varepsilon_{ABCD}, \quad (3.14)$$

$$\varepsilon_{ABCD} \eta_{DE}^{(\pm)a} = \mp (\delta_{EC} \eta_{AB}^{(\pm)a} + \delta_{EA} \eta_{BC}^{(\pm)a} - \delta_{EB} \eta_{AC}^{(\pm)a}), \quad (3.15)$$

$$\eta_{AB}^{(\pm)a} \eta_{AB}^{(\mp)b} = 0, \quad (3.16)$$

$$\eta_{AC}^{(\pm)a} \eta_{BC}^{(\pm)b} = \delta^{ab} \delta_{AB} + \varepsilon^{abc} \eta_{AB}^{(\pm)c}, \quad (3.17)$$

$$\eta_{AC}^{(\pm)a} \eta_{BC}^{(\mp)b} = \eta_{AC}^{(\mp)b} \eta_{BC}^{(\pm)a}, \quad (3.18)$$

$$\varepsilon^{abc} \eta_{AB}^{(\pm)b} \eta_{CD}^{(\pm)c} = \delta_{AC} \eta_{BD}^{(\pm)a} - \delta_{AD} \eta_{BC}^{(\pm)a} - \delta_{BC} \eta_{AD}^{(\pm)a} + \delta_{BD} \eta_{AC}^{(\pm)a} \quad (3.19)$$

where $\eta_{AB}^{(+a)} \equiv \eta_{AB}^a$ and $\eta_{AB}^{(-a)} \equiv \bar{\eta}_{AB}^a$.

Of course all these separations are due to the fact, $O(4) = SU(2)_L \times SU(2)_R$, stating that any $O(4)$ rotations can be decomposed into self-dual and anti-self-dual rotations. To be explicit, for an infinitesimal $O(4)$ transformation, i.e., $\Lambda^A_B(x) \approx \delta^A_B + \lambda^A_B(x)$, we can take the following decomposition

$$\lambda_{AB}(x) = \lambda_{(+)}^a(x) \eta_{AB}^a + \lambda_{(-)}^a(x) \bar{\eta}_{AB}^a \quad (3.20)$$

where $\lambda_{(+)}^a(x)$ and $\lambda_{(-)}^a(x)$ are local gauge parameters in $SU(2)_L$ and $SU(2)_R$, respectively. To be specific, let us introduce two families of 4×4 matrices defined by

$$[T_+^a]_{AB} \equiv \eta_{AB}^a, \quad [T_-^a]_{AB} \equiv \bar{\eta}_{AB}^a. \quad (3.21)$$

According to the definition (3.12), the matrix representation of the generators in (3.21) is given by

$$T_+^{\dot{1}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad T_+^{\dot{2}} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad T_+^{\dot{3}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad (3.22)$$

$$T_-^{\dot{1}} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad T_-^{\dot{2}} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad T_-^{\dot{3}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (3.23)$$

Then Eqs. (3.17) and (3.18) immediately show that T_{\pm}^a satisfy $SU(2)$ Lie algebras, i.e.,

$$[T_{\pm}^a, T_{\pm}^b] = -2\varepsilon^{abc}T_{\pm}^c, \quad [T_{\pm}^a, T_{\mp}^b] = 0. \quad (3.24)$$

According to the definition (3.21), the self-duality (3.13) leads to the important relation

$$[T_{\pm}^a]_{AB} = \pm \frac{1}{2} \varepsilon_{AB}{}^{CD} [T_{\pm}^a]_{CD}. \quad (3.25)$$

The 't Hooft matrices in (3.21) are two independent spin $s = \frac{3}{2}$ representations of $SU(2)$ Lie algebra. A deep geometrical meaning of the 't Hooft symbols is to specify the triple (I, J, K) of complex structures of $\mathbb{R}^4 \cong \mathbb{C}^2$ as the simplest hyper-Kähler manifold for a given orientation. The triple complex structures (I, J, K) form a quaternion which can be identified with the $SU(2)$ generators T_{\pm}^a in (3.21) [26].

Now we introduce an $O(4)$ -valued gauge field defined by $A = A^{(+a)}T_+^a + A^{(-a)}T_-^a$ where $A^{(\pm)a} = A_M^{(\pm)a} dx^M$ ($a = 1, 2, 3$) are connection one-forms on M and T_{\pm}^a are Lie algebra generators of $SU(2)_L$ and $SU(2)_R$ satisfying (3.24). The identification we want to make is then given by

$$\omega = \frac{1}{2} \omega_{AB} J^{AB} \equiv A = A^{(+a)}T_+^a + A^{(-a)}T_-^a. \quad (3.26)$$

Since the group $SO(4)$ is a direct product of normal subgroups $SU(2)_L$ and $SU(2)_R$, i.e. $SO(4) = SU(2)_L \times SU(2)_R$, we take the 4-dimensional defining representation of the Lorentz generators as follows

$$\begin{aligned} [J^{AB}]_{CD} &= \frac{1}{2} \left(\eta_{AB}^a [T_+^a]_{CD} + \bar{\eta}_{AB}^a [T_-^a]_{CD} \right) \\ &= \frac{1}{2} \left(\eta_{AB}^a \eta_{CD}^a + \bar{\eta}_{AB}^a \bar{\eta}_{CD}^a \right), \end{aligned} \quad (3.27)$$

where T_+^a and T_-^a are the $SU(2)_L$ and $SU(2)_R$ generators given by Eq. (3.21). It is then easy to check using Eqs. (3.24) and (3.19) or Eq. (3.14) that the generators in Eq. (3.27) satisfy the Lorentz algebra.

According to the identification (3.26), $SU(2)$ gauge fields can be defined from the spin connections

$$\begin{aligned}
[\omega_M]_{CD} &= \frac{1}{2}\omega_{MAB}[J^{AB}]_{CD} \\
&= \left(\frac{1}{2}\omega_{MAB}\eta_{AB}^a\right)[T_+^a]_{CD} + \left(\frac{1}{2}\omega_{MAB}\bar{\eta}_{AB}^a\right)[T_-^a]_{CD} \\
&\equiv A_M^{(+a)}[T_+^a]_{CD} + A_M^{(-a)}[T_-^a]_{CD} = [A_M]_{CD}.
\end{aligned} \tag{3.28}$$

That is, we get the decomposition (3.10) for spin connections.

Using the definition (3.21), the spin connection (3.10) and the curvature tensor (3.11) can be written as follows:

$$\omega_{MAB} = A_M^{(+a)}[T_+^a]_{AB} + A_M^{(-a)}[T_-^a]_{AB}, \tag{3.29}$$

$$R_{MNAB} = F_{MN}^{(+a)}[T_+^a]_{AB} + F_{MN}^{(-a)}[T_-^a]_{AB}, \tag{3.30}$$

where

$$F_{MN}^{(\pm)} = \partial_M A_N^{(\pm)} - \partial_N A_M^{(\pm)} + [A_M^{(\pm)}, A_N^{(\pm)}]. \tag{3.31}$$

Using the Lie algebra (3.24), one can write the field strength (3.31) as the component form

$$F_{MN}^{(\pm)a} = \partial_M A_N^{(\pm)a} - \partial_N A_M^{(\pm)a} - 2\varepsilon^{abc} A_M^{(\pm)b} A_N^{(\pm)c}, \tag{3.32}$$

which is precisely the same as Eq. (2.2). Therefore, we see that $A_M^{(\pm)} = A_M^{(\pm)a} T_\pm^a$ can be identified with $SU(2)_{L,R}$ gauge fields and $F_{MN}^{(\pm)} = F_{MN}^{(\pm)a} T_\pm^a$ with their field strengths. Indeed one can also show that the local $O(4)$ rotations in (3.2) can be represented as the gauge transformations of the $SU(2)$ gauge fields $A_M^{(\pm)}$:

$$A_M^{(\pm)} \rightarrow \Lambda_{(\pm)} A_M^{(\pm)} \Lambda_{(\pm)}^{-1} + \Lambda_{(\pm)} \partial_M \Lambda_{(\pm)}^{-1} \tag{3.33}$$

where $\Lambda_{(\pm)}(x) \equiv \exp(\lambda_{(\pm)}^a(x) T_\pm^a) \in SU(2)_{L,R}$ are group elements defined by Eq. (3.20).

Let us recall the symmetry property of curvature tensors determined by the properties about the torsion and the tangent-space group

$$R_{ABCD} = -R_{ABDC} = -R_{BACD} \tag{3.34}$$

where $R_{ABCD} = E_A^M E_B^N R_{MNCD}$. Also note that the curvature tensors satisfy the first Bianchi identity

$$R_{A[BCD]} \equiv R_{ABCD} + R_{ADBC} + R_{ACDB} = 0 \tag{3.35}$$

which is an integrability condition originated by the fact that the spin connections (3.8) are determined by potential fields, i.e., vierbeins. It is easy to see that the following symmetry can be derived by using Eqs. (3.34) and (3.35)

$$R_{ABCD} = R_{CDAB}. \tag{3.36}$$

The gravitational instantons are defined by the self-dual solution to the Einstein equation

$$R_{MNAB} = \pm \frac{1}{2} \varepsilon_{AB}^{CD} R_{MNCD}. \quad (3.37)$$

Note that a metric satisfying the self-duality equation (3.37) is necessarily Ricci-flat because $R_{MN} \equiv R_{MAN}{}^A = \pm \frac{1}{6} \varepsilon_N^{ABC} R_{M[ABC]} = 0$ and so automatically satisfies the vacuum Einstein equations (2.23). Using the decomposition (3.30) and the relation (3.25), Eq.(3.37) can be written as

$$\begin{aligned} F_{MN}^{(+a)}[T_+]_{AB} + F_{MN}^{(-a)}[T_-]_{AB} &= \pm \frac{1}{2} \varepsilon_{AB}^{CD} (F_{MN}^{(+a)}[T_+]_{CD} + F_{MN}^{(-a)}[T_-]_{CD}) \\ &= \pm (F_{MN}^{(+a)}[T_+]_{AB} - F_{MN}^{(-a)}[T_-]_{AB}). \end{aligned} \quad (3.38)$$

Therefore we should have $F_{MN}^{(-a)} = 0$ for the self-dual case with + sign in Eq. (3.37) while $F_{MN}^{(+a)} = 0$ for the anti-self-dual case with - sign and so imposing the self-duality equation (3.37) is equivalent to the half-flat equation $F^{(\pm)a} = 0$.

A solution of the half-flat equation $F^{(\pm)} = 0$ is given by $A^{(\pm)} = \Lambda_{\pm} d\Lambda_{\pm}^{-1}$ and then Eq.(3.33) shows that it is always possible to choose a self-dual gauge $A^{(\pm)a} = 0$. Therefore, one can see the following important property. If the spin connection is, for example, self-dual, i.e. $A_M^{(-)} = 0$, the curvature tensor is also self-dual, i.e. $F_{MN}^{(-)} = 0$. Conversely, if the curvature is self-dual, i.e. $F_{MN}^{(-)} = 0$, one can always choose a self-dual spin connection by a suitable gauge choice since $F_{MN}^{(-)} = 0$ requires that $A_M^{(-)}$ is a pure gauge. In other words, in this self-dual gauge, the problem of finding gravitational instantons is equivalent to one of finding self-dual spin connections [13]

$$\omega_{MAB} = \pm \frac{1}{2} \varepsilon_{AB}^{CD} \omega_{MCD} \quad (3.39)$$

which is equivalent to the (anti-)self-dual gauge condition $A_M^{(\pm)a} = 0$ according to the decomposition (3.29). The gravitational instantons defined by Eq.(3.37) are then obtained by solving the first-order differential equations defined by (3.39).

The self-duality equations (3.37) are imposed on the second group indices $[CD]$ of the curvature tensor R_{ABCD} and they do not touch the first group indices $[AB]$. But note that the first Bianchi identity (3.35) reshuffles three indices in R_{ABCD} and the symmetry (3.36) is consequently deduced. Thereby the self-duality condition for the second group should necessarily be correlated to the one for the first group [27]. In other words, because the Riemann curvature tensors satisfy the symmetry property (3.36), the gravitational instanton (3.37) is equivalent to the self-duality equation

$$R_{ABEF} = \pm \frac{1}{2} \varepsilon_{AB}^{CD} R_{CDEF}. \quad (3.40)$$

Then, using the decomposition (3.30) again, one can similarly show that the gravitational instanton (3.40) can be understood as an $SU(2)$ Yang-Mills instanton defined by (2.14), i.e.

$$F_{AB}^{(\pm)} = \pm \frac{1}{2} \varepsilon_{AB}^{CD} F_{CD}^{(\pm)} \quad (3.41)$$

where $F_{AB}^{(\pm)} = F_{AB}^{(\pm)a} T_{\pm}^a = E_A^M E_B^N F_{MN}^{(\pm)}$ are defined by Eq. (2.7). In a coordinate basis, the self-duality equation (3.41) can be written as the form (2.13) because one can deduce that

$$\begin{aligned}
E_A^M E_B^N F_{MN}^{(\pm)} &= \pm \frac{1}{2} \varepsilon_{AB}{}^{CD} E_C^P E_D^Q F_{PQ}^{(\pm)} \\
\Rightarrow F_{MN}^{(\pm)} &= \pm \frac{1}{2} \varepsilon_{AB}{}^{CD} E_M^A E_N^B E_C^P E_D^Q F_{PQ}^{(\pm)} \\
&= \pm \frac{1}{2} g_{MR} g_{NS} \varepsilon^{ABCD} E_A^R E_B^S E_C^P E_D^Q F_{PQ}^{(\pm)} \\
&= \pm \frac{1}{2} \frac{\varepsilon^{RSPQ}}{\sqrt{g}} g_{MR} g_{NS} F_{PQ}^{(\pm)}
\end{aligned} \tag{3.42}$$

where $\sqrt{g} = \det E_M^A$.

Therefore, we see that gravitational instantons defined by Eq. (3.37) are solutions of both (2.13) and (2.23) and so they can be regarded as Yang-Mills instantons in the sense that the self-duality equation of gravitational instantons can always be recast into exactly the same self-duality equation as the $SU(2)$ Yang-Mills instantons on a Ricci-flat manifold. But note that the Yang-Mills instantons as well as the four-dimensional metric used to define Eq. (3.42) are simultaneously determined by gravitational instantons. Therefore, the self-duality in Eq. (3.42) cannot be interpreted as $SU(2)$ instantons in a fixed background. Although every gravitational instantons satisfy the self-duality equation (2.13) for Yang-Mills instantons on a Ricci-flat manifold, the converse is not necessarily true: An $SU(2)$ instanton on a Ricci-flat manifold is not always a gravitational instanton. For example, Yang-Mills instantons on ALE spaces in [19, 20] and ALF spaces in [21, 22] consist of a more general class of solutions than those obtained from ALE and ALF gravitational instantons.

As was pointed out above, the self-duality in Eq. (3.42) should not be interpreted as $SU(2)$ instantons in a fixed background because we are solving the coupled equations (2.13) and (2.23). We are not solving Eq. (2.13) on a non-dynamical background manifold. Note that the Yang-Mills action (2.11) is invariant under the conformal transformation

$$g_{MN} \mapsto \tilde{g}_{MN} = \Omega^2(x) g_{MN}, \tag{3.43}$$

assuming that F_{MN} are metric-independent. As a result, the self-duality equations (2.13) are also invariant under the transformation (3.43). However the conformal transformation (3.43) is no longer a symmetry of the coupled system defined by the action (2.19) because the gravitational action (2.20) is not invariant under the transformation (3.43) and so breaks the conformal symmetry. Furthermore the assumption that F_{MN} are metric-independent is no longer valid when gravity is coupled to Yang-Mills fields. Therefore, Eq. (3.42) does not have to be invariant under the conformal transformation (3.43). Of course, this feature is consistent with the fact that the Yang-Mills instantons satisfying Eq.(3.42) are defined by the Einstein-Yang-Mills action (2.19).

We will finally check the claim that the gravitational instantons can be regarded as Yang-Mills instantons by showing that the former satisfies the same equations as the latter. First, we show that

the second Bianchi identity for curvature tensors is reduced to the Bianchi identity for $SU(2)$ gauge fields:

$$\nabla_{[M}R_{NP]AB} = 0 \quad \Leftrightarrow \quad D_{[M}^{(\pm)}F_{NP]}^{(\pm)} = 0, \quad (3.44)$$

where the bracket $[MNP] \equiv MNP + NPM + PMN$ denotes the cyclic permutation of indices. The covariant derivative on the left-hand side of Eq. (3.44) is defined by

$$\nabla_M R_{NPAB} = \partial_M R_{NPAB} - \Gamma_{MN}^Q R_{QPAB} - \Gamma_{MP}^Q R_{NQAB} - \omega_M^C{}_A R_{NPCB} - \omega_M^C{}_B R_{NPAC} \quad (3.45)$$

and, on the right-hand side, it is given by Eq. (2.18). Rewrite the covariant derivative (3.45) as the form

$$\nabla_M R_{NPAB} = \partial_M R_{NPAB} - \Gamma_{MN}^Q R_{QPAB} - \Gamma_{MP}^Q R_{NQAB} + \omega_{MAC} R_{NPCB} - R_{NPAC} \omega_{MCB}.$$

Using the decompositions (3.29) and (3.30) and the commutation relations (3.24), we get

$$\begin{aligned} \nabla_M R_{NPAB} &= \left(\partial_M F_{NP}^{(+)} - \Gamma_{MN}^Q F_{QP}^{(+)} - \Gamma_{MP}^Q F_{NQ}^{(+)} + [A_M^{(+)}, F_{NP}^{(+)}] \right)_{AB} \\ &\quad + \left(\partial_M F_{NP}^{(-)} - \Gamma_{MN}^Q F_{QP}^{(-)} - \Gamma_{MP}^Q F_{NQ}^{(-)} + [A_M^{(-)}, F_{NP}^{(-)}] \right)_{AB} \\ &= \left(D_M^{(+)} F_{NP}^{(+)} + D_M^{(-)} F_{NP}^{(-)} \right)_{AB}. \end{aligned} \quad (3.46)$$

Therefore, we arrived at the result (3.44) that the second Bianchi identity for curvature tensors is equivalent to the Bianchi identity for $SU(2)$ Yang-Mills fields. Note that all the terms containing the Levi-Civita connection in Eq.(3.44) are canceled each other.

After rewriting the self-duality equation (3.40) as

$$R_{MNAB} = \pm \frac{1}{2} \frac{\varepsilon^{RSPQ}}{\sqrt{g}} g_{MR} g_{NS} R_{PQAB}, \quad (3.47)$$

the covariant derivative is taken on both sides to yield

$$g^{PM} \nabla_P R_{MNAB} = \mp \frac{1}{2} \frac{\varepsilon_N{}^{RPQ}}{\sqrt{g}} \nabla_R R_{PQAB} = 0,$$

where the Bianchi identity (3.44) was used. The relation (3.46) then guarantees that the Yang-Mills equations

$$g^{MN} D_M^{(\pm)} F_{NP}^{(\pm)} = 0 \quad (3.48)$$

will be satisfied accordingly. So remarkably it turns out that gravitational instantons can actually be identified with Yang-Mills instantons in the sense that the gravitational and Yang-Mills instantons satisfy mathematically the same self-duality equations. But, as we discussed before, the self-duality equation (3.42) must be interpreted as self-gravitating Yang-Mills instantons rather than $SU(2)$ instantons on a rigid background.

4 Yang-Mills Instantons from Gravitational Instantons

We showed in the previous section that every gravitational instantons satisfy the self-duality equation (2.13) on a Ricci-flat manifold defined by the gravitational instanton itself. We have constructed $SU(2)$ gauge fields as the projection of the spin connection (3.2) onto the self-dual part and the anti-self-dual part by using the 't Hooft symbols. The embedding to relate gauge and spin connections was suggested long ago by Charap and Duff [27]. (See also [28].) In this section, we will elucidate with explicit examples how Yang-Mills instantons can be obtained from gravitational instantons.

To be specific, we want to find Yang-Mills instantons satisfying Eq. (2.13) where the background metric g_{MN} is a gravitational instanton obeying Eq. (3.40). First, we will calculate the spin connection (3.8) for a given gravitational instanton metric and then identify $SU(2)$ gauge fields A_M according to the identification (3.26). As was shown in (3.42), the corresponding field strength F_{MN} of the $SU(2)$ gauge fields automatically satisfies the self-duality equation (2.13) on a curved manifold M whose metric is given by the gravitational instanton itself.

We will easily reproduce already known solutions in literatures [29, 30, 31, 32] in this way. As a byproduct, we will also find new Yang-Mills instantons on a curved manifold M . It might be emphasized that it is always possible to find Yang-Mills instantons on a Ricci-flat manifold M by the same procedure whenever a gravitational instanton M is given, as will be illustrated with several examples. Here we refer to the index convention in Section 1.

4.1 Gibbons-Hawking metric

The Gibbons-Hawking metric [45] is a general class of self-dual, Ricci-flat metrics with the triholomorphic $U(1)$ symmetry which describes a particular class (A-type) of ALE and ALF instantons. The Gibbons-Hawking metric for gravitational multi-instantons is given by

$$\begin{aligned} ds^2 &= V^{-1}(x)(d\tau + q_i dx^i)^2 + V(x)dx^i dx^i \\ &\equiv e^{2\psi}(d\tau + q_i dx^i)^2 + e^{-2\psi}dx^i dx^i, \end{aligned} \quad (4.1)$$

where

$$V(x) = e^{-2\psi(x)} = \epsilon + 2m \sum_{a=1}^k \frac{1}{|x^i - x_a^i|} \quad (4.2)$$

with $\epsilon = 0$ for ALE instantons and $\epsilon = 1$ for ALF instantons. Here we use the world index $M = (i, 4 = \tau)$ with $i = 1, 2, 3$ and the frame index $A = (\hat{i}, \hat{4})$ with $\hat{i} = \hat{1}, \hat{2}, \hat{3}$. Note that $\psi = \psi(x)$, $q_i = q_i(x)$ and the Killing vector $\partial/\partial\tau$ generates the triholomorphic $U(1)$ symmetry.

One can easily read off the vierbeins from the metric (4.1) as

$$E^{\hat{4}} = e^{\psi}(d\tau + q_i dx^i), \quad E^{\hat{i}} = e^{-\psi} dx^i \quad (4.3)$$

and

$$E_{\hat{4}} = e^{-\psi} \frac{\partial}{\partial \tau}, \quad E_{\hat{i}} = e^{\psi} \left(\partial_i - q_i \frac{\partial}{\partial \tau} \right). \quad (4.4)$$

Using the torsion-free condition, $T^A = dE^A + \omega_B^A \wedge E^B = 0$, one can calculate the spin connections. For example, one can get from Eq. (4.3)

$$\begin{aligned} dE^{\hat{4}} &= e^{\psi} \left(\partial_i \psi dx^i \wedge d\tau + \partial_i \psi q_j dx^i \wedge dx^j + \frac{1}{2} f_{ij} dx^i \wedge dx^j \right) \\ &= - \left(e^{\psi} \partial_i \psi E^{\hat{4}} + \frac{1}{2} e^{3\psi} f_{ij} E^{\hat{j}} \right) \wedge E^{\hat{i}} \\ &= -\omega_{\hat{4}\hat{i}} \wedge E^{\hat{i}}, \end{aligned}$$

where $f_{ij} = \partial_i q_j - \partial_j q_i$. Therefore, one can read off

$$\omega_{\hat{4}\hat{i}} = e^{\psi} \partial_i \psi E^{\hat{4}} + \frac{1}{2} e^{3\psi} f_{ij} E^{\hat{j}}. \quad (4.5)$$

Similarly, the spin connections and the structure functions can be obtained as follows

$$\begin{aligned} \omega_{\hat{4}\hat{i}} &= e^{\psi} \partial_i \psi E^{\hat{4}} + \frac{1}{2} e^{3\psi} f_{ij} E^{\hat{j}}, \\ \omega_{\hat{i}\hat{j}} &= -\frac{1}{2} e^{3\psi} f_{ij} E^{\hat{4}} + e^{\psi} (\partial_i \psi E^{\hat{j}} - \partial_j \psi E^{\hat{i}}), \end{aligned} \quad (4.6)$$

$$\begin{aligned} f_{\hat{4}\hat{i}\hat{4}} &= -\partial_i e^{\psi}, & f_{\hat{i}\hat{j}\hat{4}} &= e^{3\psi} f_{ij}, \\ f_{\hat{4}\hat{i}\hat{j}} &= 0, & f_{\hat{j}\hat{k}\hat{i}} &= \partial_k e^{\psi} \delta_{\hat{j}}^{\hat{i}} - \partial_j e^{\psi} \delta_{\hat{k}}^{\hat{i}}. \end{aligned} \quad (4.7)$$

Note that we are explicitly discriminating the three-dimensional world and frame indices as (i, j, k, \dots) and $(\hat{i}, \hat{j}, \hat{k}, \dots)$, respectively. It is easy to see that the self-duality equation (3.39) for the spin connection (4.6) is reduced to the equation

$$\varepsilon_{\hat{i}\hat{j}\hat{k}} \partial_k \psi = \frac{1}{2} e^{2\psi} f_{ij} \Leftrightarrow \nabla V + \nabla \times \vec{q} = 0. \quad (4.8)$$

Using the result (4.8), one can now read off the self-dual $SU(2)$ gauge fields defined by $\omega_{AB} = A^a \eta_{AB}^a$:

$$\begin{aligned} A^a &= e^{2\psi} \bar{\eta}_{i\hat{4}}^a \partial_i \psi (d\tau + q_j dx^j) + \bar{\eta}_{i\hat{j}}^a \partial_i \psi dx^j \\ &= e^{\psi} \partial_i \psi \bar{\eta}_{iA}^a E^A = E_{\hat{i}} \psi \bar{\eta}_{\hat{i}A}^a E^A. \end{aligned} \quad (4.9)$$

That is, with the notation $E_{\hat{i}} \psi = e^{\psi} \partial_i \psi \equiv \partial_{\hat{i}} \psi$,

$$A_A^a = \partial_{\hat{i}} \psi \bar{\eta}_{iA}^a = \frac{1}{2} \bar{\eta}_{A\hat{i}}^a \partial_{\hat{i}} \log V. \quad (4.10)$$

It is easy to derive the following relation from Eq. (4.8)

$$e^\psi \partial_i \partial_i e^\psi - 3 \partial_i e^\psi \partial_i e^\psi = 0. \quad (4.11)$$

Using the above results, one can get the field strengths for $SU(2)$ gauge fields (4.9)

$$\begin{aligned} F_{\hat{4}\hat{i}}^a &= E_{\hat{4}} A_{\hat{i}}^a - E_{\hat{i}} A_{\hat{4}}^a - 2\varepsilon^{abc} A_{\hat{4}}^b A_{\hat{i}}^c + f_{\hat{4}\hat{i}\hat{4}} A_{\hat{4}}^a \\ &= e^\psi \partial_i \partial_a e^\psi + 3 \partial_i e^\psi \partial_a e^\psi - 2\delta_i^a \partial_k e^\psi \partial_k e^\psi, \end{aligned} \quad (4.12)$$

$$\begin{aligned} F_{\hat{i}\hat{j}}^a &= E_{\hat{i}} A_{\hat{j}}^a - E_{\hat{j}} A_{\hat{i}}^a - 2\varepsilon^{abc} A_{\hat{i}}^b A_{\hat{j}}^c + f_{\hat{i}\hat{j}\hat{4}} A_{\hat{4}}^a + f_{\hat{i}\hat{j}\hat{k}} A_{\hat{k}}^a \\ &= e^\psi \partial_k \left(\varepsilon_{a\hat{k}\hat{j}} \partial_i e^\psi - \varepsilon_{a\hat{k}\hat{i}} \partial_j e^\psi \right) - 4\varepsilon_{\hat{i}\hat{j}\hat{k}} \partial_k e^\psi \partial_a e^\psi + \partial_k e^\psi \left(\varepsilon_{a\hat{k}\hat{i}} \partial_j e^\psi - \varepsilon_{a\hat{k}\hat{j}} \partial_i e^\psi \right). \end{aligned} \quad (4.13)$$

Now it is straightforward to check that the above $SU(2)$ field strengths are self-dual, i.e.

$$F_{AB}^a = \frac{1}{2} \varepsilon_{AB}{}^{CD} F_{CD}^a. \quad (4.14)$$

To be specific, one can explicitly see that

$$\begin{aligned} \frac{1}{2} \varepsilon_{\hat{i}\hat{j}\hat{k}} F_{\hat{j}\hat{k}}^a &= -e^\psi \partial_i \partial_a e^\psi - 3 \partial_i e^\psi \partial_a e^\psi + 2\delta_i^a \partial_k e^\psi \partial_k e^\psi \\ &= F_{\hat{i}\hat{4}}^a, \end{aligned} \quad (4.15)$$

where the relation (4.11) was used. In terms of the harmonic function in Eq. (4.2), the above field strength can be represented by

$$F_{\hat{i}\hat{4}}^a = \frac{1}{2} V^{-2} \partial_i \partial_a V - \frac{3}{2} V^{-3} \partial_i V \partial_a V + \frac{1}{2} \delta_i^a V^{-3} \partial_k V \partial_k V \quad (4.16)$$

and Eq. (4.11) can be written as

$$\partial_i \partial_i \log V + \partial_i \log V \partial_i \log V = 0. \quad (4.17)$$

It would be interesting to compare Eq. (4.17) (after the replacement $\partial_i \rightarrow \partial_M$ since the function $V(x)$ does not depend on τ) with the 't Hooft ansatz $A_\mu^a = \bar{\eta}_{\mu\nu}^a \partial_\nu \log \phi(x)$ for $SU(2)$ multi-instantons (see Eq. (4.60b) in [1]) satisfying³

$$\partial_\mu \partial_\mu \log \phi + \partial_\mu \log \phi \partial_\mu \log \phi = 0. \quad (4.18)$$

Our result here recovers the self-dual gauge fields in [30] (for $H = V$).

³Note that Eq. (4.11) can be represented in terms of frame derivatives as $\partial_i \partial_i \psi - 3 \partial_i \psi \partial_i \psi = 0$ which also reduces to the form (4.18) with the identification $\psi = -\frac{1}{3} \log \phi$.

4.2 Taub-NUT metric

The Taub-NUT metric is the simplest ALF space described by the Gibbons-Hawking metric (4.1) with $\epsilon = 1$ and $k = 1$. Using the spherical coordinates, it is given by

$$ds^2 = c_r^2 dr^2 + \sum_{i=1}^3 c_i^2 (\sigma^i)^2 \quad (4.19)$$

with the coefficients $c_1 = c_2 \neq c_3$ given by

$$c_r(r) = \frac{1}{2} \sqrt{\frac{r+m}{r-m}}, \quad c_1(r) = c_2(r) = \frac{1}{2} \sqrt{r^2 - m^2}, \quad c_3(r) = m \sqrt{\frac{r-m}{r+m}}. \quad (4.20)$$

The Maurer-Cartan one-forms $\{\sigma^i\}$ satisfy the following exterior algebra [10]

$$d\sigma^i + \frac{1}{2} \varepsilon^{\hat{i}\hat{j}\hat{k}} \sigma^j \wedge \sigma^k = 0. \quad (4.21)$$

The vierbein bases are given by

$$E^{\hat{4}} = c_r dr, \quad E^{\hat{i}} = c_i \sigma^i \quad (\text{NS}[i]), \quad (4.22)$$

and

$$E_{\hat{4}} = \frac{1}{c_r} \partial_r, \quad E_{\hat{i}} = \frac{1}{c_i} \kappa_i \quad (\text{NS}[i]), \quad (4.23)$$

where κ_i are the basis vectors dual to σ^i , i.e. $\langle \sigma^i, \kappa_j \rangle = \delta_j^i$, satisfying

$$[\kappa_i, \kappa_j] = \varepsilon_{\hat{i}\hat{j}\hat{k}} \kappa_k. \quad (4.24)$$

Here we indicate no summation convention for the index i with the notation $(\text{NS}[i])$. The spin connections read as

$$\omega_{\hat{i}\hat{4}} = \frac{\partial_r c_i}{c_r} \sigma^i \quad (\text{NS}[i]), \quad \omega_{\hat{i}\hat{j}} = -\varepsilon_{\hat{i}\hat{j}\hat{k}} \frac{(c_i^2 + c_j^2 - c_k^2)}{2c_i c_j} \sigma^k \quad (\text{NS}[ij]). \quad (4.25)$$

Note that the spin connections in Eq. (4.25) are not completely self-dual, but the anti-self-dual part is simply given by $\omega_{AB}^{(-)} = \frac{1}{2} (\omega_{AB} - \frac{1}{2} \varepsilon_{AB}^{CD} \omega_{CD}) = -\bar{\eta}_{AB}^a \frac{\sigma^a}{2}$ and so their curvature tensors identically vanish thanks to Eq. (4.21). The curvature tensors are so self-dual, i.e. $R_{AB} = F^a \eta_{AB}^a$, which are given by

$$\begin{aligned} R_{\hat{1}\hat{2}} &= R_{\hat{3}\hat{4}} = \frac{8m}{(r+m)^3} \left(E^{\hat{1}} \wedge E^{\hat{2}} + E^{\hat{3}} \wedge E^{\hat{4}} \right), \\ R_{\hat{1}\hat{4}} &= R_{\hat{2}\hat{3}} = -\frac{4m}{(r+m)^3} \left(E^{\hat{1}} \wedge E^{\hat{4}} + E^{\hat{2}} \wedge E^{\hat{3}} \right), \\ R_{\hat{2}\hat{4}} &= R_{\hat{3}\hat{1}} = -\frac{4m}{(r+m)^3} \left(E^{\hat{2}} \wedge E^{\hat{4}} + E^{\hat{3}} \wedge E^{\hat{1}} \right). \end{aligned} \quad (4.26)$$

The corresponding $SU(2)$ gauge fields can be identified from (4.25) as

$$\begin{aligned} A^{\dot{1}} &\equiv \frac{1}{2}(\omega_{\dot{1}\dot{4}} + \omega_{\dot{2}\dot{3}}) = \frac{r-m}{r+m} \frac{\sigma^1}{2}, \\ A^{\dot{2}} &\equiv \frac{1}{2}(\omega_{\dot{2}\dot{4}} + \omega_{\dot{3}\dot{1}}) = \frac{r-m}{r+m} \frac{\sigma^2}{2}, \\ A^{\dot{3}} &\equiv \frac{1}{2}(\omega_{\dot{1}\dot{2}} + \omega_{\dot{3}\dot{4}}) = \left(-1 + \frac{4m^2}{(r+m)^2}\right) \frac{\sigma^3}{2}. \end{aligned} \quad (4.27)$$

Therefore, the field strength of the $SU(2)$ gauge fields (4.27) can be calculated to be

$$\begin{aligned} F &= dA + A \wedge A \\ &= \frac{1}{2} f^a(r) \eta_{AB}^a E^A \wedge E^B \end{aligned} \quad (4.28)$$

with

$$f^{\dot{1}}(r) = -\frac{4m}{(r+m)^3} = f^{\dot{2}}(r), \quad f^{\dot{3}}(r) = \frac{8m}{(r+m)^3}. \quad (4.29)$$

Note that the $SU(2)$ field strengths in (4.28) are self-dual, i.e. $F = *F$, which, of course, coincide with the curvature tensor (4.26).

Our result here agrees with the self-dual gauge fields in [31, 32].

4.3 Eguchi-Hanson metric

The Eguchi-Hanson metric [46] is the simplest ALE space described by the Gibbons-Hawking metric (4.1) with $\epsilon = 0$ and $k = 2$. Let us consider the metric given by

$$ds^2 = h^{-2}(r) dr^2 + \frac{r^2}{4}(\sigma_1^2 + \sigma_2^2) + \frac{r^2}{4} h^2(r) \sigma_3^2 \quad (4.30)$$

with the function $h(r) = \sqrt{1 - a^4/r^4}$. The Maurer-Cartan one-forms $\{\sigma^i\}$ satisfy the exterior algebra

$$d\sigma^i - \frac{1}{2} \varepsilon^{\hat{i}\hat{j}\hat{k}} \sigma^j \wedge \sigma^k = 0. \quad (4.31)$$

Note that the sign is different from the Taub-NUT case (4.21), with which the metric (4.30) becomes self-dual. The spin connections are given by Eq. (4.25) for $c_r = h^{-1}(r)$, $c_1 = c_2 = r/2$, and $c_3 = rh(r)/2$ and their components are

$$\begin{aligned} \omega_{\dot{1}\dot{2}} &= \omega_{\dot{3}\dot{4}} = \frac{1}{2} \left(1 + \frac{a^4}{r^4}\right) \sigma^3, \\ \omega_{\dot{1}\dot{4}} &= \omega_{\dot{2}\dot{3}} = \frac{1}{2} \sqrt{1 - \frac{a^4}{r^4}} \sigma^1, \\ \omega_{\dot{2}\dot{4}} &= \omega_{\dot{3}\dot{1}} = \frac{1}{2} \sqrt{1 - \frac{a^4}{r^4}} \sigma^2, \end{aligned} \quad (4.32)$$

which are clearly self-dual. The curvature tensors are straightforwardly computed by

$$\begin{aligned} R_{\hat{1}\hat{2}} &= R_{\hat{3}\hat{4}} = \frac{4a^4}{r^6} \left(E^{\hat{1}} \wedge E^{\hat{2}} + E^{\hat{3}} \wedge E^{\hat{4}} \right), \\ R_{\hat{1}\hat{4}} &= R_{\hat{2}\hat{3}} = -\frac{2a^4}{r^6} \left(E^{\hat{1}} \wedge E^{\hat{4}} + E^{\hat{2}} \wedge E^{\hat{3}} \right), \\ R_{\hat{2}\hat{4}} &= R_{\hat{3}\hat{1}} = -\frac{2a^4}{r^6} \left(E^{\hat{2}} \wedge E^{\hat{4}} + E^{\hat{3}} \wedge E^{\hat{1}} \right). \end{aligned} \quad (4.33)$$

The self-dual curvature tensors for the Eguchi-Hanson metric (4.30) can be determined by $SU(2)$ gauge fields $A^a = \frac{1}{4}\omega_{AB}\eta_{AB}^a = (f(r)\sigma^1, f(r)\sigma^2, g(r)\sigma^3)$ where

$$f(r) = \frac{1}{2}\sqrt{1 - \frac{a^4}{r^4}}, \quad g(r) = \frac{1}{2}\left(1 + \frac{a^4}{r^4}\right). \quad (4.34)$$

The corresponding $SU(2)$ field strength coincides with the curvature tensor $R_{AB} = F^a\eta_{AB}^a$ in (4.33) where $F^a = dA^a - \varepsilon^{abc}A^b \wedge A^c$ and they are given by

$$F = \frac{1}{2}f^a(r)\eta_{AB}^a E^A \wedge E^B \quad (4.35)$$

with

$$f^{\hat{1}}(r) = -\frac{2a^4}{r^6} = f^{\hat{2}}(r), \quad f^{\hat{3}}(r) = \frac{4a^4}{r^6}. \quad (4.36)$$

Our result here agrees with the self-dual gauge fields in [29, 32].

4.4 Atiyah-Hitchin metric

The Atiyah-Hitchin metric [47] describes a four-dimensional hyper-Kähler manifold with $SO(3)$ isometry that was introduced to describe the moduli space of $SU(2)$ BPS monopoles of magnetic charge 2. Let us consider the Bianchi type IX space [11] which is locally described by the metric with an $SU(2)$ or $SO(3)$ isometry group

$$ds^2 = a_\tau^2 d\tau^2 + \sum_{i=1}^3 a_i^2 (\sigma^i)^2 \quad (4.37)$$

where $a_\tau = a_1 a_2 a_3$ and a_i 's are functions solely of τ . The self-dual conditions for all Bianchi IX solutions are given by the equations

$$\begin{aligned} \frac{1}{a_\tau} \frac{da_1}{d\tau} &= \frac{a_2^2 + a_3^2 - a_1^2}{2a_2 a_3} - \alpha_1, \\ \frac{1}{a_\tau} \frac{da_2}{d\tau} &= \frac{a_3^2 + a_1^2 - a_2^2}{2a_3 a_1} - \alpha_2, \\ \frac{1}{a_\tau} \frac{da_3}{d\tau} &= \frac{a_1^2 + a_2^2 - a_3^2}{2a_1 a_2} - \alpha_3, \end{aligned} \quad (4.38)$$

where three constant numbers α_i , $i = 1, 2, 3$, satisfy $\alpha_i \alpha_j = \varepsilon_{ijk} \alpha_k$. Choosing $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 1)$ will lead to the Atiyah-Hitchin metric [47] while $(\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$ yields the Eguchi-Hanson type I or II metric [46].

Identify the vierbein basis from the metric (4.37)

$$\{E^{\hat{i}}, E^{\hat{4}}\} = \{a_i \sigma^i, a_\tau d\tau\}, \quad \{E_{\hat{i}}, E_{\hat{4}}\} = \{a_i^{-1} \kappa_i, a_\tau^{-1} \frac{\partial}{\partial \tau}\} \quad (4.39)$$

without summation convention for the index i . The left-invariant 1-forms $\{\sigma^i\}$ on S^3 satisfy the exterior algebra (4.31) and the dual basis vectors $\{\kappa_i\}$ satisfy the Lie algebra $[\kappa_i, \kappa_j] = -\varepsilon_{ijk} \kappa_k$. Note that the metric (4.37) has the same structure as the Taub-NUT metric (4.19). Therefore, the spin connections also have the same structure as follows

$$\omega_{\hat{i}\hat{4}} = \frac{a'_i}{a_\tau} \sigma^i \quad (\text{NS}[i]), \quad \omega_{\hat{i}\hat{j}} = \varepsilon_{ijk} \frac{a_i^2 + a_j^2 - a_k^2}{2a_i a_j} \sigma^k \quad (\text{NS}[ij]), \quad (4.40)$$

where the prime means the derivative with respect to τ . Note that the spin connections in (4.40) are not self-dual in general. One can check using Eq. (4.38) that the spin connections in Eq. (4.40) satisfy the following relation

$$\frac{1}{4} \bar{\eta}_{AB}^a \omega_{AB} = \frac{1}{2} \left(-\omega_{a\hat{4}} + \frac{1}{2} \varepsilon_{aj\hat{k}} \omega_{\hat{j}\hat{k}} \right) = \frac{1}{2} \alpha_a \sigma^a \quad (\text{NS}[a]). \quad (4.41)$$

Therefore, they become self-dual only when $(\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$ which was completely solved. (See Eq. (4.23) in [11] for the exact solution.) But the curvature tensors will be self-dual, i.e. $F_{MN}^{(-)} = 0$ in Eq. (3.30), because the curvature tensor of the anti-self-dual spin connections in (4.41) identically vanishes due to Eq. (4.31).

Let us define $SU(2)$ gauge fields as follows

$$A^a \equiv \frac{1}{4} \eta_{AB}^a \omega_{AB} = \omega_{a\hat{4}} + \frac{1}{2} \alpha_a \sigma^a \quad (\text{NS}[a]). \quad (4.42)$$

Our previous result (3.41) implies that the field strengths $F^a = dA^a - \varepsilon^{abc} A^b \wedge A^c$ defined by the $SU(2)$ gauge fields in (4.42) are necessarily self-dual. Now we will show that it is the case. It is straightforward to calculate the $SU(2)$ field strength

$$\begin{aligned} F^a &= \left(\frac{a'_a}{a_\tau} \right)' d\tau \wedge \sigma^a + \varepsilon^{abc} \left(\frac{a'_a}{2a_\tau} - \frac{a'_b a'_c}{a_\tau^2} - \frac{a'_b \alpha_c}{a_\tau} \right) \sigma^b \wedge \sigma^c \quad (\text{NS}[a]) \\ &= \tilde{a}'_a d\tau \wedge \sigma^a + \varepsilon^{abc} \left(\frac{\tilde{a}_a}{2} - \tilde{a}_b \tilde{a}_c - \tilde{a}_b \alpha_c \right) \sigma^b \wedge \sigma^c \quad (\text{NS}[a]), \end{aligned} \quad (4.43)$$

where $\tilde{a}_a \equiv a'_a / a_\tau$. Using the identity [48] derived from Eq. (4.38),

$$\frac{\tilde{a}'_1}{a_1 a_\tau} = -\frac{\tilde{a}_1 - 2\tilde{a}_2 \tilde{a}_3 - \tilde{a}_2 \alpha_3 - \tilde{a}_3 \alpha_2}{a_2 a_3}, \quad \text{etc}, \quad (4.44)$$

we see that the field strength (4.43) has the correct self-dual structure, i.e.

$$\begin{aligned}
F &= dA + A \wedge A \\
&= \frac{1}{2} f^a(\tau) \eta_{AB}^a E^A \wedge E^B \\
&= -\frac{\tilde{a}'_a}{2a_a a_\tau} \eta_{AB}^a E^A \wedge E^B.
\end{aligned} \tag{4.45}$$

The self-dual gauge fields in Eqs. (4.42) and (4.43) describe a Yang-Mills instanton on the Atiyah-Hitchin space and it consists of a new solution to the extent of our knowledge.

4.5 Real heaven

The real heaven metric [49] describes four dimensional hyper-Kähler manifolds with a rotational Killing symmetry which is also completely determined by one real scalar field. The metric is given by

$$\begin{aligned}
ds^2 &= (\partial_3 \psi)^{-1} (d\tau + q_\alpha dx^\alpha)^2 + (\partial_3 \psi) (e^\psi dx^\alpha dx^\alpha + dx^3 dx^3) \\
&\equiv e^{-2\phi_4} (d\tau + q_\alpha dx^\alpha)^2 + e^{2\phi_i} dx^i dx^i
\end{aligned} \tag{4.46}$$

where $q_\alpha = -\varepsilon^{\alpha\beta} \partial_\beta \psi$, ($\alpha = 1, 2$) and the function $\psi(x)$ is independent of τ and satisfy the three-dimensional continual Toda equation

$$(\partial_1^2 + \partial_2^2) \psi + \partial_3^2 e^\psi = 0. \tag{4.47}$$

The rotational Killing vector is given by $c_\alpha \partial_\alpha \psi \partial / \partial \tau$ with constants c_α .

We identify the vierbein vectors as

$$E^{\hat{i}} = e^{\phi_i} dx^i \quad (\text{NS}[i]), \quad E^{\hat{4}} = e^{-\phi_4} (d\tau + q_\alpha dx^\alpha). \tag{4.48}$$

where

$$e^{2\phi_1} = e^{2\phi_2} = \partial_3 \psi e^\psi, \quad e^{2\phi_3} = e^{2\phi_4} = \partial_3 \psi. \tag{4.49}$$

From the torsion-free equation $dE^A + \omega_B^A \wedge E^B = 0$, we get

$$\begin{aligned}
\omega_{\hat{i}\hat{4}} &= -e^{\phi_4 - \phi_i} \partial_i e^{-\phi_4} E^{\hat{4}} - \frac{1}{2} e^{-\phi_4 - \phi_i - \phi_j} f_{ij} E^{\hat{j}} \quad (\text{NS}[i]), \\
\omega_{\hat{i}\hat{j}} &= -\frac{1}{2} e^{-\phi_4 - \phi_i - \phi_j} f_{ij} E^{\hat{4}} + e^{-\phi_i - \phi_j} (\partial_j e^{\phi_i} E^{\hat{i}} - \partial_i e^{\phi_j} E^{\hat{j}}) \quad (\text{NS}[ij]),
\end{aligned} \tag{4.50}$$

where $f_{ij} = \partial_i q_j - \partial_j q_i$ with $q_i \equiv -\varepsilon^{3ij} \partial_j \psi$.

It is straightforward to check that the self dual relations, $\omega_{\hat{3}\hat{1}} = \omega_{\hat{2}\hat{4}}$ and $\omega_{\hat{2}\hat{3}} = \omega_{\hat{1}\hat{4}}$, are satisfied if and only if the continual Toda equation (4.47) is satisfied. However, the relation $\omega_{\hat{1}\hat{2}} = \omega_{\hat{3}\hat{4}}$ is not

satisfied. In order to cure this mismatch, first note that we can perform the local frame rotation (3.1) as follows

$$\begin{aligned}\tilde{E}^A &= \Lambda^A_B E^B \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \frac{\tau}{2} & -\sin \frac{\tau}{2} \\ 0 & 0 & \sin \frac{\tau}{2} & \cos \frac{\tau}{2} \end{pmatrix} \begin{pmatrix} E^{\hat{1}} \\ E^{\hat{2}} \\ E^{\hat{3}} \\ E^{\hat{4}} \end{pmatrix}.\end{aligned}\quad (4.51)$$

The spin connections also transform according to Eq.(3.2)

$$\tilde{\omega}^A_B = \Lambda^A_C \omega^C_D \Lambda^{-1D}_B + \Lambda^A_C (d\Lambda^{-1})^C_B \quad (4.52)$$

where

$$\Lambda^A_C (d\Lambda^{-1})^C_B = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} d\tau \quad (4.53)$$

and $d\tau = (-q_\alpha e^{-\phi_\alpha} E^{\hat{\alpha}} + e^{\phi_4} E^{\hat{4}})$. Note that the frame rotation (4.51) affects the self-duality condition only for $\tilde{\omega}_{\hat{3}\hat{4}} = \omega_{\hat{3}\hat{4}} + \frac{1}{2}d\tau$ due to the inhomogeneous term (4.53). In other words, $\tilde{\omega}_{\hat{3}\hat{1}} = \tilde{\omega}_{\hat{2}\hat{4}}$ and $\tilde{\omega}_{\hat{2}\hat{3}} = \tilde{\omega}_{\hat{1}\hat{4}}$ are automatically satisfied thanks to the previous relations. Now it is straightforward to check that $\tilde{\omega}_{\hat{1}\hat{2}} = \omega_{\hat{1}\hat{2}} = \tilde{\omega}_{\hat{3}\hat{4}} = (\omega_{\hat{3}\hat{4}} - \frac{1}{2}q_\alpha e^{-\phi_\alpha} E^{\hat{\alpha}} + \frac{1}{2}e^{\phi_4} E^{\hat{4}})$. Therefore, the spin connections in (4.52) become self-dual.

If one introduces $SU(2)$ gauge fields by

$$A^a \equiv \frac{1}{4} \eta^a_{AB} \tilde{\omega}_{AB} = \tilde{\omega}_{a\hat{4}} = \omega_{a\hat{4}} + \frac{1}{2} \delta^a_3 d\tau, \quad (4.54)$$

the corresponding field strengths, $F^a = dA^a - \varepsilon^{abc} A^b \wedge A^c$, should be self-dual according to the general result (3.41). This can also be proved by using the relation (3.19) which leads to the following result

$$F^a = \frac{1}{4} \eta^a_{AB} \left(d\tilde{\omega}_{AB} + \tilde{\omega}_{AC} \wedge \tilde{\omega}_{CB} \right) = \frac{1}{4} \eta^a_{AB} \tilde{R}_{AB}. \quad (4.55)$$

Hence the self-duality of F^a results from the self-dual curvature tensors \tilde{R}_{AB} . Or one can check it by a straightforward calculation using Eqs. (4.50) and (4.47) though rather tedious.

The self-dual gauge fields in Eq. (4.54) describe a Yang-Mills instanton on the real heaven (4.46), which is a new solution to the extent of our knowledge.

4.6 Euclidean Schwarzschild solution

The Euclidean Schwarzschild metric [9] was constructed by the Wick rotation of the Schwarzschild black-hole solution. It is not a gravitational instanton (not a hyper-Kähler manifold) although it is a

Ricci-flat manifold. The metric takes the form

$$ds^2 = \left(1 - \frac{2m}{r}\right) d\tau^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (4.56)$$

The radial coordinate is constrained by $r \geq 2m$ and the time coordinate τ is an angular variable with period $8\pi m$. Hence this solution has the topology $\mathbb{R}^2 \times \mathbb{S}^2$.

After defining the vierbein basis ($E^{\hat{1}} = h(r)^{-1}dr$, $E^{\hat{2}} = r d\theta$, $E^{\hat{3}} = r \sin\theta d\phi$, $E^{\hat{4}} = h(r)d\tau$), it is easy to compute spin connections:

$$\begin{aligned} \omega_{\hat{1}\hat{2}} &= -h d\theta, & \omega_{\hat{1}\hat{3}} &= -h \sin\theta d\phi, & \omega_{\hat{2}\hat{3}} &= -\cos\theta d\phi, \\ \omega_{\hat{1}\hat{4}} &= -\frac{1}{2}(h^2)'d\tau, & \omega_{\hat{2}\hat{4}} &= \omega_{\hat{3}\hat{4}} = 0, \end{aligned} \quad (4.57)$$

where $h(r) = \sqrt{1 - \frac{2m}{r}}$. The corresponding curvature tensors are given by

$$\begin{aligned} R_{\hat{1}\hat{2}} &= -\frac{m}{r^3} E^{\hat{1}} \wedge E^{\hat{2}}, & R_{\hat{1}\hat{3}} &= -\frac{m}{r^3} E^{\hat{1}} \wedge E^{\hat{3}}, & R_{\hat{1}\hat{4}} &= \frac{2m}{r^3} E^{\hat{1}} \wedge E^{\hat{4}}, \\ R_{\hat{2}\hat{3}} &= \frac{2m}{r^3} E^{\hat{2}} \wedge E^{\hat{3}}, & R_{\hat{2}\hat{4}} &= -\frac{m}{r^3} E^{\hat{2}} \wedge E^{\hat{4}}, & R_{\hat{3}\hat{4}} &= -\frac{m}{r^3} E^{\hat{3}} \wedge E^{\hat{4}}, \end{aligned} \quad (4.58)$$

which are not self-dual anymore although they are Ricci-flat, i.e., $R_{AB} \equiv R_{ACBC} = 0$.

Because the spin connections in Eq. (4.57) are neither self-dual nor anti-self-dual, we can consider both type of $SU(2)$ gauge fields defined by

$$A^{(\pm)a} \equiv \frac{1}{4} \eta_{AB}^{(\pm)a} \omega_{AB}. \quad (4.59)$$

The field strengths, $F^{(\pm)a} = dA^{(\pm)a} - \varepsilon^{abc} A^{(\pm)b} \wedge A^{(\pm)c}$, should be either self-dual (for the $+$ sign) or anti-self-dual (for the $-$ sign) because we get the following result

$$F^{(\pm)a} = \frac{1}{4} \eta_{AB}^{(\pm)a} \left(d\omega_{AB} + \omega_{AC} \wedge \omega_{CB} \right) = \frac{1}{4} \eta_{AB}^{(\pm)a} R_{AB}, \quad (4.60)$$

which can be derived by using the relation (3.19). According to the general result (3.41), the $SU(2)$ gauge fields in Eq. (4.59) automatically satisfy the self-duality equation (2.13) where the background geometry is given by the metric (4.56). Therefore, the solution (4.59) indeed describes an $SU(2)$ Yang-Mills (anti-)instanton on the space (4.56).

The solution (4.59) was originally found by Charap and Duff [27]. The reason for the revival here is that the solution (4.59) exposes an interesting structure for a Ricci-flat manifold. According to the decomposition (3.29) and (3.30), we see that the Euclidean Schwarzschild metric (4.56) describes the sum of an $SU(2)_L$ instanton and an $SU(2)_R$ anti-instanton. Therefore, an interesting question is whether this kind of feature is generic or not. Remarkably it can be shown [50] that any Einstein manifold satisfying $R_{AB} = \Lambda \delta_{AB}$ for either $\Lambda = 0$ or $\Lambda \neq 0$ always arises as the sum of $SU(2)_L$ instantons and $SU(2)_R$ anti-instantons.

5 Topological Invariants

The correspondence between gravitational and Yang-Mills instantons now raises an intriguing question about topological invariants in gravity and gauge theories. In the gravity side, there are two topological invariants associated with the Atiyah-Patodi-Singer index theorem for an elliptic complex in four dimensions [10], namely the Euler characteristic $\chi(M)$ and the Hirzebruch signature $\tau(M)$, which can be expressed as integrals of the curvature of a four dimensional metric while, in the gauge theory side, there is a unique topological invariant up to a boundary term given by the Chern class of gauge bundle. Thus a natural question is how the two kinds of topological invariants for self-dual four manifolds can be related to the Chern class of instanton bundle. In particular, the two topological invariants for gravitational instantons should be related to each other, in other words,

$$a\chi(M) + b\tau(M) = c, \quad a, b, c \in \mathbb{Z}, \quad (5.1)$$

because there is only a unique topological invariant $c_2(E)$, the second Chern class, for Yang-Mills instantons.

The topologically inequivalent sector of instanton solutions is defined by the homotopy class of a map from a three sphere at asymptotic infinity into the gauge group $G = SU(2)$

$$f : \mathbb{S}^3 \rightarrow SU(2) \quad (5.2)$$

and the topological charge is defined by an element of the homotopy group $\pi_3(SU(2)) = \mathbb{Z}$. Viewed the spin connections in Eq. (3.2) as gauge fields in $G = O(4) = SU(2)_L \times SU(2)_R$, one may also classify the topological sectors of the $O(4)$ gauge fields in Eq. (3.10) by the homotopy class of the map

$$f : \mathbb{S}^3 \rightarrow O(4) = SU(2)_L \times SU(2)_R. \quad (5.3)$$

Hence the homotopy group of $O(4)$ in the gravity theory is isomorphic to two copies of the additive group of integers

$$\pi_3(O(4)) \approx \pi_3(SU(2)_L \times SU(2)_R) \approx \mathbb{Z} \oplus \mathbb{Z}. \quad (5.4)$$

Consequently, there are two independent gravitational topological charges [10], i.e., the Euler characteristic $\chi(M)$ and the Hirzebruch signature $\tau(M)$.

The Euler number $\chi(M)$ for the de Rham complex and the signature $\tau(M)$ for the Hirzebruch

signature complex are, respectively, defined by⁴

$$\begin{aligned}\chi(M) &= \frac{1}{32\pi^2} \int_M \varepsilon^{ABCD} R_{AB} \wedge R_{CD} \\ &\quad + \frac{1}{16\pi^2} \int_{\partial M} \varepsilon^{ABCD} \left(\theta_{AB} \wedge R_{CD} - \frac{2}{3} \theta_{AB} \wedge \theta_{CE} \wedge \theta_{ED} \right),\end{aligned}\tag{5.5}$$

$$\tau(M) = -\frac{1}{24\pi^2} \int_M \text{Tr} R \wedge R - \frac{1}{24\pi^2} \int_{\partial M} \text{Tr} \theta \wedge R + \eta_S(\partial M),\tag{5.6}$$

where θ_{AB} is the second fundamental form of the boundary ∂M . It is defined by

$$\theta_{AB} = \omega_{AB} - \omega_{0AB},\tag{5.7}$$

where ω_{AB} are the actual connection 1-forms and ω_{0AB} are the connection 1-forms if the metric were locally a product form near the boundary [10]. The connection 1-form ω_{0AB} will have only tangential components on ∂M and so the second fundamental form θ_{AB} will have only normal components on ∂M . And $\eta_S(\partial M)$ is the η -function given by the eigenvalues of a signature operator defined over ∂M and depends only on the metric on ∂M [10]. The topological invariants are also related to nuts (isolated points) and bolts (two surfaces), which are the fixed points of the action of one parameter isometry groups of gravitational instantons [33].

We have verified in the previous sections that, for gravitational instantons, one of the $SU(2)$ factors in (5.3) completely decouples from the theory. Therefore, the topological classification of (anti-)self-dual spin connections will essentially be the same as Eq. (5.2) in the gauge theory. That is the reason why we expect the relation (5.1) for the topological invariants in Eqs. (5.5) and (5.6). Now we will confirm the relation (5.1) explicitly determining the coefficients.

Since θ_{AB} in Eq. (5.7) are antisymmetric on the AB index pair, we will decompose them into a self-dual part and an anti-self-dual part according to Eq. (3.10)

$$\theta_{AB} \equiv a^{(+a)} \eta_{AB}^a + a^{(-a)} \bar{\eta}_{AB}^a.\tag{5.8}$$

We take the normal to the boundary to be ($A = \hat{4}$)-direction and so we have $\theta_{\hat{4}j} = 0$. It is then straightforward to express the topological invariants in terms of $SU(2)$ gauge fields using the decompositions

⁴Note that our definition is different in signs of boundary terms from that in [10] because we choose the orientation $d^3x \wedge d\tau = -d\tau \wedge d^3x$ to be positive and the τ -direction to be normal to the boundary ∂M while the orientation $d\tau \wedge d^3x$ was chosen to be positive in [10].

(3.10), (3.11) and (5.8):

$$\begin{aligned}
\chi(M) &= \frac{1}{4\pi^2} \int_M \left(F^{(+a)} \wedge F^{(+a)} - F^{(-a)} \wedge F^{(-a)} \right) \\
&\quad + \frac{1}{4\pi^2} \int_{\partial M} \left(a^{(+a)} - a^{(-a)} \right) \wedge \left(F^{(+a)} + F^{(-a)} \right) \\
&\quad + \frac{1}{12\pi^2} \int_{\partial M} \varepsilon^{abc} \left(a^{(+a)} - a^{(-a)} \right) \wedge \left(a^{(+b)} - a^{(-b)} \right) \wedge \left(a^{(+c)} - a^{(-c)} \right) \\
&= \frac{1}{16\pi^2} \int_M \sqrt{g} \varepsilon^{ABCD} \left(F_{AB}^{(+a)} F_{CD}^{(+a)} - F_{AB}^{(-a)} F_{CD}^{(-a)} \right) d^4x \\
&\quad + \frac{1}{8\pi^2} \int_{\partial M} \sqrt{h} \varepsilon^{\hat{i}\hat{j}\hat{k}} \left(a_i^{(+a)} - a_i^{(-a)} \right) \left(F_{\hat{j}\hat{k}}^{(+a)} + F_{\hat{j}\hat{k}}^{(-a)} \right) d^3x \\
&\quad + \frac{1}{12\pi^2} \int_{\partial M} \sqrt{h} \varepsilon^{abc} \varepsilon^{\hat{i}\hat{j}\hat{k}} \left(a_i^{(+a)} - a_i^{(-a)} \right) \left(a_j^{(+b)} - a_j^{(-b)} \right) \left(a_k^{(+c)} - a_k^{(-c)} \right) d^3x, \quad (5.9) \\
\tau(M) &= \frac{1}{6\pi^2} \int_M \left(F^{(+a)} \wedge F^{(+a)} + F^{(-a)} \wedge F^{(-a)} \right) \\
&\quad + \frac{1}{12\pi^2} \int_{\partial M} \left(a^{(+a)} - a^{(-a)} \right) \wedge \left(F^{(+a)} - F^{(-a)} \right) + \eta_S(\partial M) \\
&= \frac{1}{24\pi^2} \int_M \sqrt{g} \varepsilon^{ABCD} \left(F_{AB}^{(+a)} F_{CD}^{(+a)} + F_{AB}^{(-a)} F_{CD}^{(-a)} \right) d^4x \\
&\quad + \frac{1}{24\pi^2} \int_{\partial M} \sqrt{h} \varepsilon^{\hat{i}\hat{j}\hat{k}} \left(a_i^{(+a)} - a_i^{(-a)} \right) \left(F_{\hat{j}\hat{k}}^{(+a)} - F_{\hat{j}\hat{k}}^{(-a)} \right) d^3x + \eta_S(\partial M), \quad (5.10)
\end{aligned}$$

where we defined the volume forms as $E^{\hat{1}} \wedge E^{\hat{2}} \wedge E^{\hat{3}} \wedge E^{\hat{4}} \equiv \sqrt{g} d^4x$ and $E^{\hat{1}} \wedge E^{\hat{2}} \wedge E^{\hat{3}}|_{\partial M} \equiv \sqrt{h} d^3x$.

An interesting pattern appears in the topological invariants. First consider a compact Einstein manifold without boundary, i.e. $\partial M = 0$. It turns out [50] that $F^{(+a)}$ and $F^{(-a)}$ are self-dual and anti-self-dual instantons, respectively. Then we see that the Euler number $\chi(M) = \chi^+(M) + \chi^-(M)$ does not distinguish self-dual and anti-self-dual instantons since both contribute with equal sign while the Hirzebruch signature $\tau(M) = \tau^+(M) - \tau^-(M)$ distinguishes self-dual and anti-self-dual instantons. Based on the observation, we can draw general properties about 4-dimensional compact Einstein manifolds where all boundary terms vanish. As we mentioned above, the Euler number $\chi(M)$ gets equal sign contributions from self-dual and anti-self-dual gauge fields while the Hirzebruch signature $\tau(M)$ is not the case. Thus we see that $\chi(M) \geq 0$ with the equality only if M is flat. This is the Berger's result [10]. We can further refine the Berger's result by looking at the expressions (5.9) and (5.10):

$$\chi(M) - \frac{3}{2} \tau(M) = -\frac{1}{2\pi^2} \int_M F^{(-a)} \wedge F^{(-a)} \geq 0 \quad (5.11)$$

because $F^{(-)}$ describes $SU(2)$ anti-instantons. The inequality (5.11) will be saturated if and only if a compact four-manifold is half-flat, i.e. $F^{(-a)} = 0$. In the result, we get a general relation

$$\chi(M) \geq \frac{3}{2} |\tau(M)| \quad (5.12)$$

where the bound is saturated only for \mathbb{T}^4 and $K3$ surface, which are compact self-dual four-manifolds as either trivial or nontrivial gravitational instantons. This result is known as the Hitchin-Thorpe inequality [10].

For noncompact manifolds, there are additional boundary terms as shown in (5.9) and (5.10) which are not separated into the self-dual and anti-self-dual parts unlike as the volume terms. In particular, the eta-invariant $\eta_S(\partial M)$ for k self-dual gravitational instantons [35] is given by

$$\eta_S(\partial M) = -\frac{2\epsilon}{3k} + \frac{(k-1)(k-2)}{3k} \quad (5.13)$$

where $\epsilon = 0$ for ALE boundary conditions and $\epsilon = 1$ for ALF boundary conditions. Because the topological invariants for a noncompact manifold with boundary have nontrivial boundary corrections, it is not easy to demonstrate the relation (5.1) although such a relation should exist for general half-flat manifolds. But, one may infer by investigating known examples so far that the following relation

$$\chi(M) = |\tau(M)| + 1 \quad (5.14)$$

would be satisfied for noncompact gravitational instantons. It turns out [33, 34, 35] that ALE instantons including all ADE series and ALF instantons of AD series satisfy the relation (5.14).⁵

Therefore, the evidence for the relation (5.14) is overwhelming. Since we believe that the relation (5.1) will be generic independently of asymptotic boundary conditions and topology, we conjecture that the relation (5.14) will be true for general noncompact gravitational instantons. It may be proved by showing the following identity for gravitational instantons, e.g., with $F^{(-)a} = 0$ and so taking the self-dual gauge $A^{(-)a} = 0$:

$$\begin{aligned} \chi(M) - \tau(M) &= \frac{1}{12\pi^2} \int_M F^{(+a)} \wedge F^{(+a)} + \frac{1}{6\pi^2} \int_{\partial M} a^{(+a)} \wedge F^{(+a)} \\ &\quad + \frac{1}{12\pi^2} \int_{\partial M} \varepsilon^{abc} a^{(+a)} \wedge a^{(+b)} \wedge a^{(+c)} - \eta_S(\partial M) \\ &= 1. \end{aligned} \quad (5.15)$$

Indeed, for ALE and ALF spaces, one can derive the relation $\chi(M) - \tau(M) = 1 - 4I_{\frac{1}{2}}(S_{\pm}, D)$ using Eqs. (12), (13), (14) and (20) in [34]. If M has a spin structure, the index of the Dirac operator, $I_{\frac{1}{2}}(S_{\pm}, D)$, must identically vanish [11], and thus we confirm the above identity. For general cases, we do not know how to rigorously prove the above identity and so we leave it as our conjecture.

The topological invariant in $SU(2)$ gauge theory is given by the second Chern number

$$k = \frac{1}{16\pi^2} \int_M F_{YM}^a \wedge F_{YM}^a \quad (5.16)$$

⁵ A_{k-1} ALE ($\epsilon = 0$) and ALF ($\epsilon = 1$) instantons are described by the Gibbons-Hawking metric (4.1) and D_0 ALF instantons are described by the Atiyah-Hitchin metric (4.37). Especially, Kronheimer obtained the explicit construction of the ALE manifolds as hyper-Kähler quotients [15] which heavily relies on the algebraic structure of the Kleinian groups Γ and the crucial identification between the Hirzebruch signature $\tau(M)$ and the number of conjugacy classes of the finite group Γ . See the Table 2 in [51] for the relation (5.14) of all ALE manifolds. See also the Table D.1 in [10].

where $F_{YM}^a = dA_{YM}^a + \frac{1}{2}\varepsilon^{abc}A_{YM}^b \wedge A_{YM}^c$. Note that the $SU(2)$ field strength coming from the spin connections is given by $F_G^a = dA_G^a - \varepsilon^{abc}A_G^b \wedge A_G^c$. So they are related by $A_{YM}^a = -2A_G^a$ and $F_{YM}^a = -2F_G^a$ [32]. Taking this factor into account, one can see that the Chern number (5.16) has the same normalization factor as the Euler number in Eq. (5.9), i.e.,

$$k = \frac{1}{4\pi^2} \int_M F_G^a \wedge F_G^a. \quad (5.17)$$

This fact provides us an interesting insight why the instanton number (5.17) for $SU(2)$ instantons satisfying (2.13) is not necessarily integer-valued [29, 30]. Note that the Euler number (5.5) as well as the signature (5.6) are all integer-valued. Therefore, if there is a nontrivial boundary correction in the Euler number (5.9), the instanton number (5.17) will not be an integer, i.e., a fractional number in general. We will illustrate it with explicit examples.

5.1 Taub-NUT space

For the product metric

$$ds^2 = \frac{1}{4} \frac{r_0 + m}{r_0 - m} dr^2 + \frac{1}{4} (r_0^2 - m^2) (\sigma_1^2 + \sigma_2^2) + m^2 \frac{r_0 - m}{r_0 + m} \sigma_3^2, \quad (5.18)$$

the spin connections are given by

$$(\omega_0)_{\hat{i}\hat{4}} = 0, \quad (\omega_0)_{\hat{i}\hat{j}} = \omega_{\hat{i}\hat{j}}(r = r_0). \quad (5.19)$$

Hence the second fundamental form at the boundary $r = r_0$ is

$$\theta_{\hat{i}\hat{4}} = \omega_{\hat{i}\hat{4}}(r = r_0), \quad \theta_{\hat{i}\hat{j}} = 0 \quad (5.20)$$

or

$$a^{\hat{1}} = \frac{r_0}{r_0 + m} \sigma^1, \quad a^{\hat{2}} = \frac{r_0}{r_0 + m} \sigma^2, \quad a^{\hat{3}} = \frac{2m^2}{(r_0 + m)^2} \sigma^3. \quad (5.21)$$

Using Eqs. (4.28) and (5.21), we get the following result

$$\begin{aligned} F^a \wedge F^a &= 24m^3 \frac{r - m}{(r + m)^5} \sigma^1 \wedge \sigma^2 \wedge \sigma^3 \wedge dr, \\ a^a \wedge F^a|_{r=r_0} &= -4m^2 \frac{(r_0 - m)^2}{(r_0 + m)^4} \sigma^1 \wedge \sigma^2 \wedge \sigma^3, \\ a^{\hat{1}} \wedge a^{\hat{2}} \wedge a^{\hat{3}} &= \frac{2m^2 r_0^2}{(r_0 + m)^4} \sigma^1 \wedge \sigma^2 \wedge \sigma^3. \end{aligned} \quad (5.22)$$

Therefore, we see that the boundary integrals vanish because

$$a^a \wedge F^a|_{r_0 \rightarrow \infty} = 0, \quad a^{\hat{1}} \wedge a^{\hat{2}} \wedge a^{\hat{3}}|_{r_0 \rightarrow \infty} = 0. \quad (5.23)$$

Finally we get the topological numbers for the Taub-NUT space

$$\begin{aligned}\chi(M) &= \frac{1}{4\pi^2} \int_M F^a \wedge F^a \\ &= \frac{24m^3}{4\pi^2} \underbrace{\int_{\mathbf{S}^3} \sigma^1 \wedge \sigma^2 \wedge \sigma^3}_{=16\pi^2} \underbrace{\int_m^\infty \frac{r-m}{(r+m)^5} dr}_{=\frac{1}{96m^3}} = 1,\end{aligned}\tag{5.24}$$

$$\begin{aligned}\tau(M) &= \frac{1}{6\pi^2} \int_M F^a \wedge F^a + \eta_S(\partial M) \\ &= \frac{2}{3} + \eta_S(\partial M) = 0.\end{aligned}\tag{5.25}$$

We have used the result (5.13) for the η -invariant with $k = 1$. In this case, the Euler number (5.24) is equal to the instanton number (5.17) because there is no boundary correction [31, 32]. And it is straightforward to check the relation (5.15).

5.2 Eguchi-Hanson space

For the product metric

$$ds^2 = \left(1 - \frac{a^4}{r_0^4}\right)^{-1} dr^2 + \frac{r_0^2}{4}(\sigma_1^2 + \sigma_2^2) + \frac{r_0^2}{4}\left(1 - \frac{a^4}{r_0^4}\right)\sigma_3^2,\tag{5.26}$$

the second fundamental form at the boundary $r = r_0$ is

$$a^{\dot{1}} = \frac{1}{2}\sqrt{1 - \frac{a^4}{r_0^4}}\sigma^1, \quad a^{\dot{2}} = \frac{1}{2}\sqrt{1 - \frac{a^4}{r_0^4}}\sigma^2, \quad a^{\dot{3}} = \frac{1}{2}\left(1 + \frac{a^4}{r_0^4}\right)\sigma^3.\tag{5.27}$$

Note that we have to choose the angular coordinate ranges

$$0 \leq \theta < \pi, \quad 0 \leq \varphi < 2\pi, \quad 0 \leq \psi < 2\pi\tag{5.28}$$

to remove the apparent singularities in the metric at $r = a$. Thus the boundary at ∞ becomes $\mathbb{R}P^3$.

Then we obtain the following result

$$\begin{aligned}F^a \wedge F^a &= \frac{6a^8}{r^9} \sigma^1 \wedge \sigma^2 \wedge \sigma^3 \wedge dr, \\ a^a \wedge F^a|_{r_0 \rightarrow \infty} &= 0, \\ a^{\dot{1}} \wedge a^{\dot{2}} \wedge a^{\dot{3}}|_{r_0 \rightarrow \infty} &= \frac{1}{8} \sigma^1 \wedge \sigma^2 \wedge \sigma^3,\end{aligned}\tag{5.29}$$

and get the topological numbers for the Eguchi-Hanson space

$$\begin{aligned}
\chi(M) &= \frac{1}{4\pi^2} \int_M F^a \wedge F^a + \frac{1}{12\pi^2} \int_{\partial M} \varepsilon^{abc} a^a \wedge a^b \wedge a^c \\
&= \frac{6a^8}{4\pi^2} \underbrace{\int_{\mathbf{RP}^3} \sigma^1 \wedge \sigma^2 \wedge \sigma^3}_{=8\pi^2} \underbrace{\int_a^\infty \frac{1}{r^9} dr}_{=\frac{1}{8a^8}} + \frac{6}{96\pi^2} \underbrace{\int_{\mathbf{RP}^3} \sigma^1 \wedge \sigma^2 \wedge \sigma^3}_{=8\pi^2} \\
&= \frac{3}{2} + \frac{1}{2} = 2,
\end{aligned} \tag{5.30}$$

$$\tau(M) = \frac{1}{6\pi^2} \int_M F^a \wedge F^a + \eta_S(\partial M) = 1. \tag{5.31}$$

Unlike the Taub-NUT case, there is a nontrivial boundary correction for the Euler number (5.30). Since the instanton number (5.17) does not take the boundary contribution into account, it gets a fractional number $k = \frac{3}{2}$ [30, 32]. One can check that the relation (5.15) is satisfied.

5.3 Euclidean Schwarzschild solution

This solution is interesting because it has a nontrivial Euler number [9] although it is not a gravitational instanton. But it turns out that this solution is actually the sum of $SU(2)_L$ instanton and $SU(2)_R$ anti-instanton, which explains why it has a nontrivial Euler number.

Take the product metric

$$ds^2 = \left(1 - \frac{2m}{r_0}\right) d\tau^2 + \left(1 - \frac{2m}{r_0}\right)^{-1} dr^2 + r_0^2 (d\theta^2 + \sin^2 \theta d\phi^2). \tag{5.32}$$

The second fundamental form at the boundary $r = r_0$ is then given by

$$a^{\dot{1}} = -\frac{m}{r_0^2} d\tau, \quad a^{\dot{2}} = a^{\dot{3}} = 0. \tag{5.33}$$

Using the result (4.58) with the definition $F^{(\pm)a} = \frac{1}{4} \eta_{AB}^{(\pm)a} R_{AB}$, we obtain

$$\begin{aligned}
F^{(\pm)a} \wedge F^{(\pm)a} &= \pm \frac{3m^2}{r^4} dr \wedge d\Omega \wedge d\tau, \\
a^{(\pm)a} \wedge F^{(\pm)a}|_{r_0 \rightarrow \infty} &= 0, \\
a^{(\pm)\dot{1}} \wedge a^{(\pm)\dot{2}} \wedge a^{(\pm)\dot{3}} &= 0.
\end{aligned} \tag{5.34}$$

It is then straightforward to get the topological invariants [9]

$$\chi(M) = \chi^+(M) + \chi^-(M) = 2, \tag{5.35}$$

$$\tau(M) = \tau^+(M) - \tau^-(M) = 0, \tag{5.36}$$

where $\chi^+(M) = \chi^-(M) = 1$ and $\tau^+(M) = \tau^-(M) = \frac{2}{3} + \eta(\partial M)$. Hence we confirm that the Euclidean Schwarzschild solution (4.56) is the sum of an $SU(2)$ instanton and an anti-instanton. And the relation (5.14) implies that $\tau^\pm(M) = 0$ or $\eta(\partial M) = -\frac{2}{3}$. Therefore the $SU(2)$ instanton for the Euclidean Schwarzschild solution (4.56) has the same topological invariants as the Taub-NUT space (4.19) [27]. Note that two instantons belong to different gauge groups, one in $SU(2)_L$ and the other in $SU(2)_R$, and so they cannot decay into a vacuum. As a result, the space (4.56) should be stable at least perturbatively. One may ask whether this kind of feature is special or general. Remarkably it can be shown [50] that any Ricci-flat four-manifold always arises as the sum of $SU(2)_L$ instantons and $SU(2)_R$ anti-instantons. Hence any Ricci-flat manifold should be stable for the same reason.

5.4 Topological invariant of Yang-Mills instantons

We have noticed that the instanton number (5.16) for (anti-)self-dual gauge fields satisfying (2.13) is not necessarily integer-valued because it does not take possible boundary corrections into account. But the equivalence of the self-dual systems in (2.13) and (3.37) implies that we need to also consider boundary contributions for the topological charge of Yang-Mills instantons defined on a curved manifold. Thereby we suggest the Chern number for an instanton bundle including boundary corrections

$$k = \frac{1}{16\pi^2} \int_M F^a \wedge F^a + \frac{1}{16\pi^2} \int_{\partial M_\infty} A^a \wedge F^a - \frac{1}{96\pi^2} \int_{\partial M_\infty} \varepsilon^{abc} A^a \wedge A^b \wedge A^c \quad (5.37)$$

which can be identified with the Euler characteristic (5.9) in the self-dual gauge, $A^{(-)a} = 0$, with the gauge theory normalization $A_{YM}^a = -2A_G^a$ and $F_{YM}^a = -2F_G^a$ and is accordingly integer-valued. Note that the boundary term in (5.37) is precisely the Chern-Simons form for the $SU(2)$ vector bundle at an asymptotic infinity.

Now we consider the four-manifold M to have two ends, one at an asymptotic infinity ∂M_∞ and the other at an inner boundary ∂M_0 describing nuts and bolts of gravitational instantons [33]. For example, the inner boundary is at $r = m$ for the Taub-NUT space (4.19) and at $r = a$ for the Eguchi-Hanson space (4.30). Using the identity $F^a \wedge F^a = dK$ where

$$K = A^a \wedge dA^a + \frac{1}{3} \varepsilon^{abc} A^a \wedge A^b \wedge A^c \quad (5.38)$$

and the boundary operation $\partial M = \partial M_0 - \partial M_\infty$,⁶ one can rewrite the instanton number (5.37) as the Chern-Simons integral on the inner boundary ∂M_0 , i.e.,

$$k = \frac{1}{16\pi^2} \int_{\partial M_0} \left(A^a \wedge F^a - \frac{1}{6} \varepsilon^{abc} A^a \wedge A^b \wedge A^c \right). \quad (5.39)$$

Recall that the instanton number (5.37) is simply the expression of the Euler number (5.9) and the Euler number $\chi(M)$ can be determined by the set of nuts and bolts through the fixed point theorem (Eq. (4.6) in [33])

$$\chi(M) = \sharp(\text{nuts}) + 2 \sharp(\text{bolts}). \quad (5.40)$$

⁶The sign is due to our choice of orientation. See the footnote 4.

Then we get a very interesting result that the Chern-Simons integral (5.39) on the inner boundary ∂M_0 simply counts the number of nuts plus the twice of the number of bolts in gravitational instantons:

$$k = \frac{1}{16\pi^2} \int_{\partial M_0} \left(A^a \wedge F^a - \frac{1}{6} \varepsilon^{abc} A^a \wedge A^b \wedge A^c \right) = \sharp(\text{nuts}) + 2 \sharp(\text{bolts}). \quad (5.41)$$

It is easy to check the result (5.41) for the Taub-NUT space ($\sharp(\text{nuts}) = 1$, $\sharp(\text{bolts}) = 0$) and for the Eguchi-Hanson space ($\sharp(\text{nuts}) = 0$, $\sharp(\text{bolts}) = 1$), using the previous results with the relation $A_{YM}^a = -2A_G^a$ and $F_{YM}^a = -2F_G^a$.

6 Discussion

Let us go back to the questions we have raised in Section 1. So far we have focused on the similarity between gauge theory and gravitation. A main source of the similarity is coming from the fact that the $O(4)$ -valued 1-forms $\omega^A{}_B$ are gauge fields (a connection of the spin bundle SM) with respect to $O(4)$ rotations as shown in Eq. (3.2). Then the Riemann curvature tensors in (3.7) constitute $O(4)$ -valued curvature 2-forms of the spin bundle SM . Therefore, the four-dimensional Euclidean gravity can be formulated as a gauge theory using the language of the $O(4)$ gauge theory. Via the fact that the Lorentz group $O(4)$ is a direct product of normal subgroups $SU(2)_L$ and $SU(2)_R$, i.e. $O(4) = SU(2)_L \times SU(2)_R$, the four-dimensional Euclidean gravity can be decomposed into two copies of $SU(2)$ gauge theories. In particular, the (anti-)self-dual sector satisfying (3.37) can be formulated as an $SU(2)$ gauge theory, as clearly indicated in Eq. (3.33).

Nevertheless, gravity is different from gauge theory in many aspects. A decisive source of the difference is the existence of a Riemannian metric which does not have any counterpart in gauge theory. We highlight some crucial differences between gauge theory and gravitation with the following table:

Property	Einstein	Yang-Mills
Metric	$g_{MN}(x)$ or E^A	...
Torsion	$dE^A + \omega^A{}_B \wedge E^B = 0$...
Cyclic identity	$R^A{}_B \wedge E^B = 0$...
Einstein equation	$G_{MN} = 8\pi G T_{MN}$...
Coupling constant	$[G] = L^2$	$[g_{YM}] = L^0$
Symmetry	Spacetime	Internal
Interaction	Long-range	Short-range

The metric is constrained to be covariantly constant with respect to the Levi-Civita connection (3.9) or equivalently the vierbeins are constrained to be torsion-free, i.e., $T^A = dE^A + \omega^A{}_B \wedge E^B = 0$. This constraint leads to the result that the spin connections $\omega^A{}_B$ are determined by potential fields, i.e., vierbeins, as Eq. (3.8). As a result, a primary field for gravity is the metric tensor rather than a gauge

field (a connection of vector bundle). This extra structure comprises a core origin of the differences in the above table.

Recently one of us showed [25] (see also recent reviews [26] and [52]) that Einstein gravity can be derived from electromagnetism in noncommutative space. In particular, the vierbeins E_A in gravity arise from the leading order of noncommutative $U(1)$ gauge fields and higher order terms give rise to derivative corrections to Einstein gravity. Actually the Einstein equations arising from the noncommutative gauge fields and the resulting emergent gravity motivate to newly address the questions in Section 1 in a more broad context to include noncommutative $U(1)$ gauge theories. For example, it was rigorously shown in [53] that noncommutative $U(1)$ instantons are equivalent to gravitational instantons. Therefore, it will be very interesting to find a precise map between noncommutative $U(1)$ instantons and Yang-Mills instantons because a particular class of Yang-Mills instantons can be obtained from gravitational instantons as was shown in this paper. We hope to draw some valuable insights from this line of thought in our future works.

Now our method in Section 3 can easily be generalized to get new instanton solutions by the conformal rescaling method [38]. Suppose that (M, g) is a self-dual gravitational instanton and consider a Weyl transformation given by Eq. (3.43) which can be represented as $\tilde{E}^A = \Omega(x)E^A \in \Gamma(T^*M)$ or $\tilde{E}_A = \Omega^{-1}(x)E_A \in \Gamma(TM)$ in terms of vierbeins. Under the Weyl transformation, the spin connections transform as follow:

$$\tilde{\omega}_{AB} = \omega_{AB} + (E_B \log \Omega E^A - E_A \log \Omega E^B). \quad (6.1)$$

We can apply the decompositions (3.10) and (3.11) to the transformed spin connection (6.1) and the corresponding curvature tensor $\tilde{R} = d\tilde{\omega} + \tilde{\omega} \wedge \tilde{\omega}$, respectively. After all, we will get new $SU(2)$ gauge fields defined by

$$\tilde{A}^{(+a)} = A^{(+a)} + \mathfrak{A}^{(+a)}, \quad \tilde{A}^{(-a)} = \mathfrak{A}^{(-a)} \quad (6.2)$$

where $A^{(+a)}$ are the self-dual gauge fields determined by the original self-dual spin connection ω_{AB} and

$$\mathfrak{A}^{(\pm)a} \equiv \frac{1}{2} \eta_{AB}^{(\pm)a} (E_B \log \Omega) E^A \quad (6.3)$$

and the corresponding $SU(2)$ field strengths will be given by

$$\tilde{F}^{(\pm)a} = d\tilde{A}^{(\pm)a} - \varepsilon^{abc} \tilde{A}^{(\pm)b} \wedge \tilde{A}^{(\pm)c}. \quad (6.4)$$

Now we can make two different choices:⁷

$$(I) \quad \tilde{F}_{AB}^{(-a)} = \frac{1}{2} \varepsilon_{AB}{}^{CD} \tilde{F}_{CD}^{(-a)}, \quad (6.5)$$

$$(II) \quad \tilde{F}_{AB}^{(-a)} = -\frac{1}{2} \varepsilon_{AB}{}^{CD} \tilde{F}_{CD}^{(-a)}. \quad (6.6)$$

⁷It may be worthwhile to compare the solution (6.3) with 't Hooft ansatz (see Sect. 4.3. in [1]) in singular (the case (I)) and regular (the case (II)) gauges. Note that the solution (4.9) from the Gibbons-Hawking metric takes the form (6.3) for the case (I).

For the first choice (I), we will get a self-dual Yang-Mills instanton while, for the second choice (II), an anti-self-dual Yang-Mills instanton. Then one can show [50] that, for the case (I), the Ricci-scalar $\tilde{R} = \tilde{g}^{MN} \tilde{R}_{MN}$ will identically vanish, i.e. $\tilde{R} = 0$, but the case (II) seems to give rise to an intriguing manifold satisfying $\tilde{R}_{MN} - \frac{1}{4} \tilde{g}_{MN} \tilde{R} = 0$. Because the Ricci scalar transforms under the Weyl transformation (3.43) as $\Omega^3 \tilde{R} = \Omega R - 6 \square_g \Omega$ where \square_g refers to the scalar Laplacian on (M, g) , we see that the rescaling function $\Omega(x)$ must be harmonic, i.e. $\Omega^{-1} \square_g \Omega = 0$, for the case (I), taking into account that $R = 0$. But the harmonic function $\Omega(x)$ will allow mild singularities [38] which can be removed by a gauge transformation.

By the same procedure as Eq. (3.42), the self-dualities in Eqs. (6.5) and (6.6) can be written as

$$\tilde{F}_{MN}^{(-)} = \pm \frac{1}{2} \frac{\varepsilon^{RSPQ}}{\sqrt{\tilde{g}}} \tilde{g}_{MR} \tilde{g}_{NS} \tilde{F}_{PQ}^{(-)} \quad (6.7)$$

where $\sqrt{\tilde{g}} = \Omega^4 \sqrt{g}$. However, taking into account the conformal invariance of self-duality, we get the self-duality equation on the original four-manifold (M, g) , i.e.,

$$\tilde{F}_{MN}^{(-)} = \pm \frac{1}{2} \frac{\varepsilon^{RSPQ}}{\sqrt{g}} g_{MR} g_{NS} \tilde{F}_{PQ}^{(-)}. \quad (6.8)$$

Consequently, we get new Yang-Mills instantons on an original Ricci-flat manifold (M, g) after the Weyl transformation (6.1). More details about explicit solutions obtained in this way and their topological properties will be discussed elsewhere.

In this paper we showed that any gravitational instanton is an $SU(2)$ Yang-Mills instanton on the gravitational instanton itself. Regarding to this property, there is an interesting theorem (Example 3 (page 302) in [19] and see also Sect. 7 in [20]) that there always exists an instanton bundle on an ALE manifold M with the instanton number $k = 1 - \frac{1}{|\Gamma|}$ ($|\Gamma|$ denoting the order of Γ in $M \cong \mathbb{C}^2/\Gamma$) defined by (5.16) such that the moduli space of self-dual connections on the instanton bundle is a four-dimensional hyper-Kähler manifold and coincides with the base manifold M . Inferred from our result, the above property seems to be true for other self-dual manifolds. To be precise, suppose that $\mathcal{M}(E \rightarrow M, k)$ is the moduli space of self-dual connections on a vector bundle E over M with instanton number k where M is a gravitational instanton. Then, each non-empty, non-compact 4-dimensional component of the moduli space $\mathcal{M}(E \rightarrow M, k)$ is isomorphic to the gravitational instanton itself. It will be interesting to clarify this assertion.

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